NUMERICAL SOLUTION FOR NONLINEAR MIXED PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT: *In this paper, numerical methods have been used to solve non-linear Partial Differential Equations (PDE's).Solution of non-linear PDE's involving mixed partial differential have been discussed using Laplace Decomposition Method(LDM), Adomian Decomposition Method (ADM) and Homotopy Perturbation Method (HPM) respectively. Some examples of non-linear PDE's involving mixed Partial derivatives is solved using above mentioned methods to explain the methods and to check the similarity of the results. Graphical representations of the results are also given.*

Keywords: Non-linear Mixed Partial Differential Equations, Laplace Decomposition Method, Adomian Decomposition Method, Homotopy Perturbation Method

1. INTRODUCTION

Nonlinear ordinary or partial differential equations involving mixed partial derivatives arise in various fields of science, physics and engineering. The Method of separation of variables [1] and Variational iteration method [2-4] has been extensively worked out for many years by numerous authors. Starting from the pioneer ideas of the Inokuti -Se kine- Mura method [5], JI-Huan He [4] developed the Variational iteration method (VIM) in 1999.For example, T.A.Abassy. et al [6,7] also proposed further treatments of these modification results by using Pade approximants and the Laplace transform.

An advantage of the decomposition method is that it can provide analytical approximation to a rather wide class of nonlinear equations without linearization, perturbation, closure approximations or discretization methods which can result in massive numerical computation. Recently a great deal of interest has been focused on the application of Adomian's decomposition method [19] to solve a wide variety of stochastic and deterministic problems [8].Although the Adomian's goal is to find a method to unify linear and nonlinear, ordinary or partial differential equations for solving initial and boundary value problems.

In the recent decade, several scholars in the fields of partial differential equations have paid attention in showing the existence and the solutions of the class of partial differential equations involving mixed and non mixed derivatives. Several methods were proposed, for instance, the Laplace transform method [9–11], the Mellin transform method [12], the Fourier transform method [13,14], and the Sumudo transform method [15-17] and the Green function method [18] for linear cases. Perturbation method, variational iteration method [20-22], homotopy decomposition and perturbation method [23-26], and others were developed for both linear and nonlinear cases.

2. MATERIAL AND METHODS

Many problems in natural and engineering sciences are modelled by Partial differential equations (PDE's). After studying the Laplace decomposition method that was applied to solve some examples which were nonlinear partial differential equations involving Partial Derivatives. Now the basic motivation of the present paper is the implementation of two methods on the same examples, which are Adomian decomposition method and Homotopy perturbation method. In these methods we obtain the same exact or approximate solutions. Results of all three methods are shown by graphically and analytically.

2.1 Laplace Decomposition Method

We suppose the general form of non-homogeneous partial differential equation with initial conditions as given below:

$$Lu(x,y) + Ru(x,y) = h(x,y) \quad (1)$$
$$u(x,0) = f(x)$$
$$u_y(0,y) = g(y)$$

where $L = \frac{\partial}{\partial x \partial y}$, Ru(x,y) is the rest of linear terms in which there is only first order partial derivatives of u(x,y) with respect to either x or y and h(x,y) is the basis term. We can write the equation (1) in the following form:

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) + Ru(x,y) = h(x,y) \quad (2)$$

Putting $\frac{\partial u}{\partial y} = U$ in equation (2), we get

$$\frac{\partial U}{\partial x} + Ru(x,y) = h(x,y) \quad (3)$$

Applying Laplace transform on equation (3) respecting x, we get

$$U(s,y) = \frac{1}{s}g(y) + \frac{1}{s}L_x[h(x,y) - Ru(x,y)] \quad (4)$$

Applying inverse Laplace transform of equation (4) respecting x, we get

$$U(x,y) = g(y) + L_x^{-1}\big[L_x[h(x,y) - Ru(x,y)]\big] \quad (5)$$

Again putting the value of U(x, y) in equation (5), we get

$$\frac{\partial u(x,y)}{\partial y} = g(y) + L_x^{-1}\big[L_x[h(x,y) - R(x,y)]\big] \quad (6)$$

This is the first order partial differential equation in the variables x and y. Captivating the Laplace transform of equation (6) respecting y, we get

$$u(x,s) = \frac{1}{s}f(x) + \frac{1}{s}L_y\left[g(y) + L_x^{-1}\left[\frac{1}{s}L_x[h(x,y) - Ru(x,y)]\right]\right] \quad (7)$$

Captivating the inverse Laplace transform of equation (7) respecting y, we get

$$u(x,s) = f(x) + L_y^{-1}\left[\frac{1}{s}L_y\left[g(y) + L_x^{-1}\left[\frac{1}{s}L_x[h(x,y) - Ru(x,y)]\right]\right]\right]$$

This equation gives us the exact solution of initial value problem.

2.2 Adomian Decomposition Method

Let L_x and L_t representing x and y respectively. To be specific here, we prefer $L_x = \frac{\partial^2}{\partial x^2}$ and $L_t = \frac{\partial}{\partial t}$.

Suppose we have also expression $L_t L_x u = \frac{\partial^3 u}{\partial t \partial x^2}$ and consider

$$Fu = L_t u + L_x u + L_t L_x u = g(x,t) \quad (8)$$

with initial conditions specified. The inverse operators L_t^{-1} and L_x^{-1} are defined integrations from 0 to t and 0 to x, correspondingly, for the initial value problem.

Hence $L_t^1 L_t u = u(x,t) - u(x,0)$and

$$L_x^{-1} L_x u = u(x,t) - u(0,t) - x u_x(0,t). \quad (9)$$

Solving above equation for the two linear terms thus

$$L_t u = g - L_x u - L_t L_x u,$$
$$L_x u = g - L_t u - L_t L_x u.$$

Operate on the first equation with L_t^{-1} and on the other hand with L_x^{-1}. We get

$$u = u(x,0) + L_t^{-1} g - L_t^{-1} L_x u - L_x u - (L_x u)|_{t=0,}$$
$$u = u(0,t) + x u_x(0,t) + L_x^{-1} g - L_x^{-1} L_t u - L_t u + (L_t u)|_{x=0}.$$

Adding and dividing by 2,

$$u(x,t) = \frac{1}{2}\{u(x,0) + u(0,t) + x u_x(0,t) - u_x(x,0) + u_t(0,t) + [L_t^{-1} + L_x^{-1}]g - [L_t^{-1} L_x + L_x^{-1} L_t]u(x,t) - [L_x + L_t]u(x,t)\}.$$

We identifying the expressions relating initial conditions and the in-homogenous term g as u_0 in the decomposition $u = \sum_{n=0}^{\infty} u_n$.Thus

$$u_0 = \frac{1}{2}\{u(x,0) + u(0,t) + x u_x(0,t) - u_x(x,0) + u_t(0,t) + [L_t^{-1} + L_x^{-1}]g\}.$$

where $u_{n+1} = -\frac{1}{2}\{L_t^{-1} L_x + L_x^{-1} L_t - L_x - L_t\} u_n$

For $n \geq 0$

2.3 Homotopy Perturbation Method

We devote this section to the discussion undertaking the general method to derive the special solution of

$$\partial_{x^n}^n \partial_{y^m}^m \ldots \partial_{t^i}^i [U(x,y,\ldots,t)] + L[U(x,y,\ldots,t)] + N[U(x,y,\ldots,t)] = f(x,y,\ldots,t), \quad (10)$$

We will assume that H $(x,, \ldots, t)$ is the solution of the linear part of ; we can record an illustration to appropriate the value of the selected singular point, for example, at $X(x,y,\ldots..t)$and then the corrected solution can be written as follows:

$$U(\alpha,\beta,\ldots,\tau) = H(\alpha,\beta,\ldots,\tau) + \int_0^{\alpha} \ldots \int_0^{\tau} \lambda(x,y,\ldots,t) \times \left(\partial_{x^n}^n \partial_{y^m}^m \ldots \partial_{t^i}^i [U(x,y,\ldots,t)] + L[U(x,t,\ldots t)] + N[U(x,y,\ldots,t)] - f(x,y,\ldots,t)\right) dx \ldots dt. \quad (11)$$

We will point out that U$(x, y, \ldots, t)$ is the Lagrange multiplier [3] and the second expression on the right is called the modification. The method has been modified into an iteration method [4–8] in the subsequent approach:

$$U_{n+1}(\alpha,\beta,\ldots,\tau) = H(\alpha,\beta,\ldots,\tau) + \int_0^{\alpha} \ldots \int_0^{\tau} \lambda(x,y,\ldots,t) \times \left(\partial_{x^n}^n \partial_{y^m}^m, \ldots, \partial_{t^i}^i [U_n(x,y,\ldots,t)] + L[U(x,y,\ldots,t)] + N[\bar{U}(x,y,\ldots,t)] - f(x,y,\ldots,t)\right) dx \ldots dt. \quad (12)$$

Besides $H(\alpha,\beta, \ldots, \tau)$ as preliminary guesstimate with likely nonentities and $\bar{U}(x,y, \ldots, t)$ is pondered as a circumscribed adaptation meaning $\delta\bar{U}(x,y, \ldots, t) = 0$. Indeed for random $(\alpha, \beta,\ldots,)$, the above equation can be reformulated as follows:

$$U_{n+1}(X,Y,\ldots,T) = H(X,Y,\ldots,T) + \int_0^{X} \ldots \int_0^{T} \lambda(x,y,\ldots,t) \times \left(\partial_{x^n}^n \partial_{y^m}^m, \ldots, \partial_{t^i}^i [U_n(x,y,\ldots,t)] + L[U(x,y,\ldots,t)] + N[\bar{U}(x,y,\ldots,t)] - f(x,y,\ldots,t)\right) dx \ldots dt. \quad (13)$$

3. NUMERICAL RESULTS

Here we discuss solutions of PDE's with mixed partial derivatives using LDM, ADM and HPM respectively.

Example 1

Suppose the partial differential equation

$$\frac{\partial^2 u}{\partial x \partial y} = e^{-y} \cos x \quad (14)$$

by means of initial condition $u(x,0) = 0$, $u_y(0,y) = 0$

3.1 Solution using LDM

In the given initial value problem $Lu(x,y) = \frac{\partial^2 u}{\partial x \partial y}$, $h(x,y) = e^{-y} \cos x$ and general linear term $Ru(x,y)$ is zero. Writing the equation (14) in the following form

Putting $\frac{\partial u}{\partial y} = U$ in equation, we get $\frac{\partial U}{\partial x} = e^{-y} \cos x$ (15)

It is the non-homogenous partial differential equation of first order. Taking Laplace transform on both sides of equation (15) respecting x,

$$U(s,y) = e^{-y}.\frac{1}{1+s^2} \quad (16)$$

taking inverse Laplace transform on equation (16) respecting x, we get

$$U(x,y) = e^{-y}.\cos x.$$

Now we have to solve

$$\frac{\partial U(x,y)}{\partial y} = e^{-y} \cos x \quad (17)$$

It is the partial differential equation of first order in the variables x & y. Taking Laplace transform on equation (17) respecting y, we get

$$Su(x,s) - 0 = \sin x.\frac{1}{s(1+s)} \quad (18)$$

Taking inverse Laplace transform on equation (18) respecting y, we get

$$u(x,y) = \sin x.(1 - e^{-y})$$

3.2 Solution using ADM

Define $L_1(.) = \frac{\partial}{\partial x}$

Apply above condition the equation becomes,

$$L_1\left(\frac{\partial u}{\partial y}\right) = e^{-y} \cos x \quad (19)$$

Now applying Laplace operator,

$$u_y(x,y) = L_1^{-1} e^{-y} \cos x$$

Again define $L_2(.) = \frac{\partial}{\partial y}$ using the condition

Applying Laplace operator,

$$u(x,y) = L_2^{-1} L_1^{-1} e^{-y} \cos x$$

The recursive relation is as follows:

$$u_{n+1}(x,y) = L_2^{-1} L_1^{-1} e^{-y} \cos x$$
$$u_1(x,y) = (1 - e^{-y}) \sin x$$

3.3 Solution Using HPM

Applying the convex Homotopy method on the equation (14), we obtain

$$\frac{\partial^2 u_0}{\partial x \partial y} + p\left(\frac{\partial^2 u_1}{\partial x \partial y}\right) + p^2\left(\frac{\partial^2 u_2}{\partial x \partial y}\right) + \cdots = e^{-y} \cos x$$

$$u_0 + p u_1 + p^2 u_2 + \cdots = \int_0^y \int_0^x e^{-y} \cos x \, dt$$

And comparing the coefficients of like powers of p of above

$$u(x,y) = (1 - e^{-y}) \sin x$$

Table 1: Summary of results obtained in Example 1

X	Y	$u = (1 - e^{-y}) \sin x$
0	0	0
0.1	0.1	0.009500
0.2	0.2	0.036012

0.3	0.3	0.076593
0.4	0.4	0.128383
0.5	0.5	0.188639
0.6	0.6	0.254560
0.7	0.7	0.324308
0.8	0.8	0.395027
0.9	0.9	0.464850
1	1	0.531911

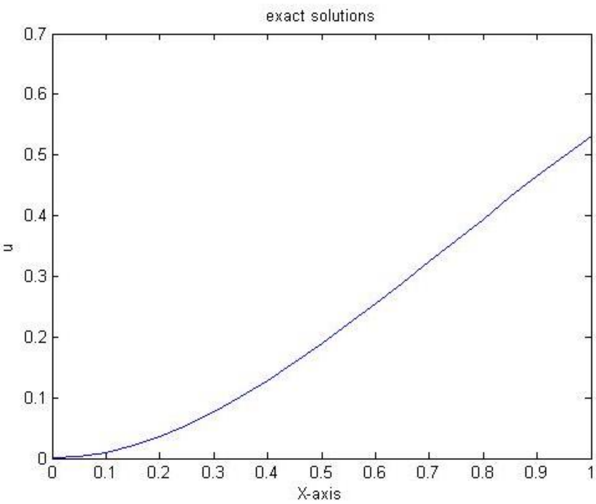

Figure: 1 Graph of Example 1

Example 2

Suppose the partial differential equation

$$\frac{\partial^2 u}{\partial y \partial x} = \sin x \sin y \qquad (20)$$

by the initial conditions

$u(x,0) = 1 + \cos x\,,\ u_y(0,y) = -2\sin y$

3.4 Solution using LDM

Suppose that $u_x(x,y)$ and $u_y(x,y)$ both are differentiable in the domain of definition of function $u(,y)$ [Young's Theorem]. It implies that $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$. Given initial condition force to write the equation in following form and putting $\frac{\partial u}{\partial y} = U$

$$\frac{\partial U}{\partial x} = \sin x \sin y$$

Taking Laplace transform on equation respecting x, we get

$$U(x,y) = -\frac{2\sin y}{s} + \sin y\left[\frac{1}{s} - \frac{s}{1+s^2}\right] \qquad (21)$$

Applying inverse Laplace transform of equation (21) with respect to x, we get

$$\frac{\partial u(x,y)}{\partial x} = -2\sin y + \sin y\,(1-\cos x) \qquad (22)$$

Applying Laplace transform and then applying inverse Laplace transform of equation (22) respecting y,

$$u(x,y) = (1+\cos x)\cos y$$

3.5 Solution using ADM

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \sin x \sin y$$

Define $L_1(.) = \frac{\partial}{\partial x}$

Apply above condition, the equation becomes,

$$L_1 u_y = \sin x \sin y$$

Apply inverse Laplace operator,

$$u_y = -2\sin y - \cos x \sin y + \sin y$$

Again define $L_2(.) = \frac{\partial}{\partial y}$

Apply above condition, the equation (20) becomes,

$L_2 u(x,y) = -2\sin y - \cos x \sin y + \sin y$

Apply Laplace operator,

$$u(x,y) = \cos x \cos y + \cos y$$

3.6 Solution using HPM

Applying the convex Homotopy method on the equation (20), we obtain

$$\frac{\partial^2 u_0}{\partial y \partial x} + p\left(\frac{\partial^2 u_1}{\partial y \partial x}\right) + p^2\left(\frac{\partial^2 u_2}{\partial y \partial x}\right) + \cdots = \sin x \sin y$$

$u_0 + pu_1 + p^2 u_2 + \cdots = 1 + \cos x + 2\cos y - 2 + \int_0^x \int_0^y (\sin x \sin y)\,dt$

And comparing the coefficients of like powers of p of above equation we get

$$p^{(0)}: u_0(x,y) = \cos y\,(1+\cos x)$$

$$p^{(1)}: u_1(x,y) = 0$$

$$p^{(2)}: u_2(x,y) = 0, \ldots\ldots\ldots\ldots..$$

$$u(x,y) = (1+\cos x)\cos y$$

Table 2: Summary of results obtained in example 2

X	**Y**	$\boldsymbol{u = (1+\cos x)\cos y}$
0	0	2
0.1	0.1	1.985037
0.2	0.2	1.940595
0.3	0.3	1.868003
0.4	0.4	1.769412
0.5	0.5	1.647732
0.6	0.6	1.506513
0.7	0.7	1.349825
0.8	0.8	1.182105
0.9	0.9	1.008007
1	1	0.832228

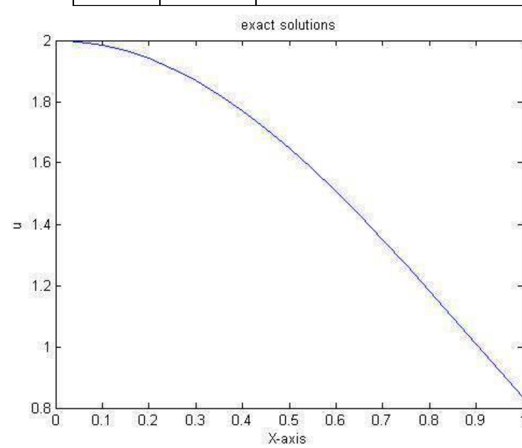

Figure: 2 Graph of example: 2

EXAMPLE 3

Suppose the subsequent partial differential equation with $Ru(x,y) \neq 0$

$$\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial x} + u = 6x^2 y \qquad (23)$$

with initial conditions

$u(x,0) = 1\,, u(0,y) = y\,, u_y(0,y) = 0$

3.7 Solution using LDM

In the above example $Ru(x,y) = \frac{\partial^2 u}{\partial x \partial y} + u(x,y)$. Here we put $\frac{\partial u}{\partial y} = U(x,y)$in equation (22), we get

$$\frac{\partial U}{\partial x}+\frac{\partial u}{\partial x}+u=6x^2y \quad (24)$$

Applying Laplace transform of equation (24) respecting x, we get

$$U(s,y)=-u(s,y)+\frac{y}{s}-\frac{1}{s}L_x[u(x,y)]+\frac{12y}{s^4} \quad (25)$$

Applying inverse Laplace transform of equation (25) respecting x, we get

$$\frac{\partial u(x,y)}{\partial x}=-u(x,y)+y-L_x^{-1}\left[\frac{1}{s}L_x\{u(x,y)\}\right]+2x^3y \quad (26)$$

Taking Laplace transform of equation (26) with respect to y, we get

$$u(x,s)=\frac{1}{s}-\frac{1}{s}L_y\left[u(x,y)+L_x^{-1}\left\{\frac{1}{s}L_x\,.\,u(x,y)\right\}\right]+\frac{1}{s^3}+2x^3.\frac{1}{s^2} \quad (27)$$

Applying inverse Laplace transform of equation (27) respecting y, we get

$$u(x,y)=1-L_y^{-1}\left[\frac{1}{s}L_y\left[u(x,y)+L_x^{-1}\left\{\frac{1}{s}L_x\,.(x,y)\right\}\right]\right]+\frac{y^2}{2!}+x^3y^2$$

Here recursive relation is given by,

$$u_{n+1}(x,y)=1-L_y^{-1}\left[\frac{1}{s}L_y\left[u_n(x,y)+L_x^{-1}\left\{\frac{1}{s}L_x u_n(x,y)\right\}\right]\right]+\frac{y^2}{2!}+x^3y^2$$

$u_0(x,y)=1$

$u_1(x,y)=-y-xy+\frac{y^2}{2!}+x^3y^2$

$u_2(x,y)=y^2+xy^2-\frac{y^3}{6}-\frac{x^3y^3}{3}+\frac{x^2y^2}{4}-\frac{xy^3}{6}-\frac{x^4y^3}{12}+x^3y^2$

3.8 Solution using ADM

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial x}(u)+u=6x^2y$$

Define $L_1(.)=\frac{\partial}{\partial x}$

Apply above condition, equation (23) becomes,

$$L_1\left(\frac{\partial u}{\partial y}\right)+L_1u+u=6x^2y \quad (28)$$

Apply Laplace operator we get,

$L_1^{-1}(.)=\int_0^x(.)dx$

$u_y(x,y)+u(x,y)-y+L_1^{-1}u=L_1^{-1}6x^2y$

Define $L_2(.)=\frac{\partial}{\partial y}$

Apply above condition, the equation (28) becomes

$L_2u(x,y)+u(x,y)-y+L_1^{-1}u=L_1^{-1}(6x^2y)$

Apply Laplace operator,

$$L_2^{-1}(.)=\int_0^y(.)dy$$

$u(x,y)=1-L_2^{-1}u+\frac{y^2}{2}-L_2^{-1}L_1^{-1}u+x^3y^2$

The recursive relation is given by,

$u_{n+1}(x,y)=1-L_2^{-1}u_n+\frac{y^2}{2}-L_2^{-1}L_1^{-1}u_n+x^3y^2$

$u_0=1$

Let $f_1=\frac{y^2}{2}+x^3y^2$

$u_1=-L_2^{-1}u_0-L_2^{-1}L_1^{-1}u_0+f_1$

$u_1(x,y)=\frac{y^2}{2}+x^3y^2-y-xy$

$u_2(x,y)=\frac{y^2}{2}+x^3y^2-L_2^{-1}u_1-L_2^{-1}L_2^{-1}u_1$

$u_2(x,y)=y^2+x^3y^2-\frac{y^3}{6}-\frac{x^3y^3}{3}-\frac{xy^3}{6}-\frac{x^4y^3}{12}+xy^2+\frac{x^2y^2}{4}$

3.9 Solution using HPM

Apply the convex Homotopy method on the equation ,

$$\frac{\partial^2u_0}{\partial x\partial y}+p\frac{\partial^2u_1}{\partial x\partial y}+p^2\frac{\partial^2u_2}{\partial x\partial y}+\cdots=-p\left(\frac{\partial u_0}{\partial x}+p\frac{\partial u_1}{\partial x}+p^2\frac{\partial u_2}{\partial x}+\cdots\right)-p(u_0+pu_1+p^2u_2+\cdots)+6x^2y$$

$$u_0+pu_1+p^2u_2+\cdots=1-p\int_0^y\int_0^x\left(\frac{\partial u_0}{\partial x}+p\frac{\partial u_1}{\partial x}+p^2\frac{\partial u_2}{\partial x}+\cdots\right)dxdy-p\int_0^y\int_0^x(u_0+pu_1+p^2u_2+\cdots)dtdt+\int_0^y\int_0^x(6x^2y)dxdy$$

And comparing the coefficients of like powers of p of the equation

$p^{(0)}: u_0=1$

$P^{(1)}: u_1(x,y)=-y+\frac{y^2}{2}-xy+x^3y^2$

$p^{(2)}: u_2(x,y)=y^2-\frac{y^3}{6}+xy^2-\frac{x^3y^3}{3}-\frac{xy^3}{6}+\frac{x^2y^2}{4}-\frac{x^4y^3}{12}+x^3y^2$

4. CONCLUSION

In this paper, Laplace decomposition method is applied to solve non-linear partial differential equations involving mixed Partial Derivatives using the initial values. Two examples are solved to explain the method. These examples are also solved by Adomian decomposition method and Homotopy perturbation method. In all these methods we obtain the same exact or approximate solutions. The results of these examples tell us that all these methods can be used alternatively for the solution of Higher-order initial value applications.

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