# A SIMPLE PROOF OF $A_5$ AS A SUBGROUP OF $S_7$ BY USING HARACTER TABLE OF $S_7$

**A. Mahboob[1], T. Hussain[3], I. Ali[3]**
Department of Mathematics, University Of Education Lahore, (Okara Campus) Pakistan
Department of Mathematics, University of Lahore, Pakistan
aabidmahboob@gmail.com, shergarhcity@gmail.com, iftikharali462@gmail.com

***ABSTRACT:*** *In this paper, the existence of alternating group $A_5$ as a subgroup of sym- metric group Sn, for n = 7 is proved by using character table of $S_7$.*

**Key Words:** Symmetric Group, Character Table and Triangular Group.

**1 INTRODUCTION**: The symmetric group Sn is defined over the regular figure n-gon with order n! for n=7. It has fifteen conjugacy classes corresponding to partition P(7).discused in [1]. The symmetric group S7 is however a non simple group of order 24 .32 .5.7 but it contains a simple derived subgroup A7 of index 2 of the order 23 .32 .5.7. by [5], A5 is the smallest non abelian simple group of order 22 .3.5 and the smallest non solvable group containing four conjugacy classes. A group Δ(2,3,k) of the form Δ(2,3,k)= < x,y;x2 =y3 =(xy)k =1 >

Where k is any positive integer is known as triangular group [6]. For k = 5 it is isomorphic to the alternating group A5 . We know by [4], Every finite alternating and symmetric group except A6 ,A7 ,A8 ,S5 ,S6 ,S7 and S8 is a factor group of Δ(2,3,k).

In this paper, we focus on a proof of the following theorem.

**Theorem:** Let G be symmetric group of degree 7. Then there exist non conjugate classes of simple groups isomorphic to $A_5$ involved by classes $C\alpha, C\beta$ and $C\gamma$. such that $\alpha^2 = \beta^2 = 1 = (\alpha\ \beta) = \gamma^5$ in $S_7$.

**Proof:**

Since | S7 | = 24 .32 .5.7 is divisible by | A5 | = 22 .3.5

Therefore by Lagrange's Theorem A5 may be a candidate to exist within S7 as a subgroup. In order to search for the possibility of the existence of A5 within S7 , we need to know necessary information about conjugancy classes and character table of S7. It is a fact that A5 is the smallest non-abelian simple group and isomorphic to Δ(2,3,k) which is generated by elements x of order 2, y of order 3 and (xy)5 =1. Δ(2,3,5)= < x, y ; x2 =y3 =(xy)5 =1 >. By [3], If α and β are two class representations of classes C α and C β of the order 2 and order 3 respectively and their product αβ = γ is an element of the only class of order 5 in S7 , then

$$\#<\alpha.\beta> = \frac{|S_7|}{|C_G(\alpha)|\,|C_G(\beta)|}\sum_{1}^{15}.\frac{\chi_i(\alpha)\ \chi_i(\beta)\ \overline{\chi_i(\gamma)}}{\chi_i(1)}$$

where #< α.β > gives number of solutions of equations α.β = γ (known as class constants) χi (1) = the degree of characters of G. χi (α) and χi (β) stands for characters values of ith character of corresponding conjugacy classes and $\overline{\chi i(\gamma)}$ is conjugate of the character value of class representation of γ.

It is noted from the character table S7 . Given on page (2).

There are three classes of the elements of order 2 of the permutation type 2α = (12), 2β =(12)(34) ,2γ =(12)(34)(56). There are two classes of the elements of order three of the permutation type 3α = (123), 3β = (123)(456). There is only one class of the element of order 5 of the permutation type 5α = (12345).

By the use of character table of S7 the following table of class coefficient reveals that only even permutations are involved in existence of Δ(2,3,5).

In the below tables we can see that first table is character table and second table is of conjugacy classes.

α=(12)(34), β=(123)(456), γ=(12345) does generate in the construction of (2,3,5). Hence for each of the relation #< α.β = γ > = 10 there exist A5 = < x,y;x2 =y3 =(xy)5 = 1 >. We find that #< α.β = γ > = 10. We observe that |CG (5)| = 10, and all these 10 relations are conjugate by conjugating x and y both by the elements of the centralizer CG (5) of an element of order 5. Since each (2,3,5) relation generate an A5 therefore 1 of such A5 are generated within S7 by different conjugates x and y. That is if x is of order 2 and y is of order 3 in S7 with in their specified corresponding conjugacy classes. Then xy is of order 5. Now the only question is how many of such A5 are conjugate?

As |CG (5)|=10

| Class | (1) | (2; 1) | ($2^2$; 1) | ($2^3$; 1) | (3; 1) | (3; 2; 1) | (3; $2^2$) |
|---|---|---|---|---|---|---|---|
| n(l) | 1 | 21 | 105 | 105 | 70 | 420 | 210 |
| $\chi_1$ | 1 | -1 | 1 | -1 | 1 | -1 | 1 |
| $\chi_2$ | 6 | 4 | 2 | 0 | 3 | 1 | -1 |
| $\chi_3$ | 14 | 6 | 2 | 2 | 2 | 0 | 2 |
| $\chi_4$ | 15 | 5 | -1 | -3 | 3 | -1 | -1 |
| $\chi_5$ | 14 | 4 | 2 | 0 | -1 | 1 | -1 |
| $\chi_6$ | 35 | 5 | -1 | 1 | -1 | -1 | -1 |
| $\chi_7$ | 20 | 0 | -4 | 0 | 2 | 0 | 2 |
| $\chi_8$ | 21 | 1 | 1 | -3 | -3 | 1 | 1 |
| $\chi_9$ | 21 | -1 | 1 | 3 | -3 | -1 | 1 |
| $\chi_{10}$ | 35 | -5 | -1 | -1 | -1 | 1 | -1 |
| $\chi_{11}$ | 15 | -5 | -1 | 3 | 3 | 1 | -1 |
| $\chi_{12}$ | 14 | -4 | 2 | 0 | -1 | -1 | -1 |
| $\chi_{13}$ | 14 | -6 | 2 | -2 | 2 | 0 | 2 |
| $\chi_{14}$ | 6 | -4 | 2 | 0 | 3 | -1 | -1 |
| $\chi_{15}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

| $(3^2; 1)$ | (4; 1) | (4,2,1) | (4,3) | (5,1) | (5,2) | (6,1) | (7) |
|---|---|---|---|---|---|---|---|
| 280 | 210 | 630 | 420 | 504 | 504 | 840 | 720 |
| 1 | -1 | 1 | -1 | 1 | -1 | -1 | 1 |
| 0 | 2 | 0 | -1 | 1 | -1 | 0 | -1 |
| -1 | 0 | 0 | 0 | -1 | 1 | -1 | 0 |
| 0 | 1 | -1 | 1 | 0 | 0 | 0 | 1 |
| 2 | -2 | 0 | 1 | -1 | -1 | 0 | 0 |
| -1 | -1 | 1 | -1 | 0 | 0 | 1 | 0 |
| 2 | 0 | 0 | 0 | 0 | 0 | 0 | -1 |
| 0 | -1 | -1 | -1 | 1 | 1 | 0 | 0 |
| 0 | 1 | -1 | 1 | 1 | -1 | 0 | 0 |
| -1 | 1 | 1 | 1 | 0 | 0 | -1 | 0 |
| 0 | -1 | -1 | -1 | 0 | 0 | 0 | 1 |
| 2 | 2 | 0 | -1 | -1 | 1 | 0 | 0 |
| -1 | 0 | 0 | 0 | -1 | -1 | 1 | 0 |
| 0 | -2 | 0 | 1 | 1 | 1 | 0 | -1 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

| | β=(123), $\lvert C_G(\beta)\rvert$=72 | β=(123)(456), $\lvert C_G(\beta)\rvert$= 18 | |
|---|---|---|---|
| α=(12), $\lvert C_G(\alpha)\rvert$=240 | 0 | 0 | $\lvert C_G(\gamma)\rvert$ =10 |
| α=(12)(34), $\lvert C_G(\alpha)\rvert$=48 | 5 | 10 | $\lvert C_G(\gamma)\rvert$ =10 |
| α=(12)(34)(56), $\lvert C_G(\alpha)\rvert$=48 | 0 | 0 | $\lvert C_G(\gamma)\rvert$ =10 |

Where γ is a cyclic group of order 5. As α and β are two elements of order 2 and 3 in S7 with in their specified conjugacy classes and γ = α.β is of order 5 then for any element c of order 5 in CG (5) We have

$c(\alpha.\beta)c-1 = c\,\gamma\, c-1 = \gamma$

This implies

$c\ \alpha\ c^{-1}\ c\ \beta\ c^{-1} = \gamma$

thus $(c\ \alpha\ c^{-1})\ (c\ \beta\ c^{-1}) = \gamma$

The same γ is produced by conjugating α and β with an element c of order 5 in CG (γ). So total number of pairs of α and β shall count to be equal in number to order of the centralizer of γ. Thus all the 10 relations that we obtained using character table become conjugate by conjugating α and β by the centralizer of γ. This concludes that all copies of A5 stand conjugate to each other, which shall form a single conjugate class within S7 .

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