INTRA-REGULAR ORDRED TERNARY∗-SEMIGROUP

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ABSTRACT: *Using the concept of intra-regular ordered semigroup studied by Chong-Yin Wu [1], we introduce in this paper, the notion of ternary intra-regular ordered semigroup. Many properties valid for the binary case can be proved for the ternary case. Our characterization of ternary intra-regular semigroup is partial, a complete characterization still open, is the converse of the proposition 3.6 true? Another question is: if S is intra-regular, is R = I in the theorem 3.7? For some needs we have introduced some new notions "Strong and weakly ideals generated by an element,.. The principal references are those used by [1], however we add some other references.*

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1.Preliminaries

Definition 1.1. Let S be a non-empty set endowed with a ternary operation ":", S is called a ternary semigroup if

$\forall\, a,b,c,e \in S, a.b.(c.d.e) = a.(b.c.d).e = (a.b.c).d.e$

In the sequel $a.b.c$ will be just denoted abc.

Definition1.2.

1. An ordered ternary semigroup S is a partial ordering set at the same time a ternary semigroup such that for any $a,b,x,y \in S, a \leq b$ implies $xay \leq xby$, $axy \leq bxy$ and $xya \leq xyb$.
2. An ordered ternary semigroup S with a unary operation $*: S \longrightarrow S$ is called an ordered ternary $*-$semigroup if it satisfies $(x)^* = x$ and $(xyz)^* = z^*y^*x^*$ for any $x,y,z \in S$. Such a unary operation $*$ is called an involution. If for any a,b with $a \leq b$, we have $a^* \leq b^*$, then $*$ is called an order preserving involution.

Example1.3. Let $S = \{a,b,c,d,e\}$ be a set on which we define a ternary operation ":" by the table below and an order " $\leq$ " by

$\leq = \{(a,a),(a,b),(b,b),(c,c),(d,b),(d,c),(d,d),(e,c),(e,e)\}$.

$(S,.,\leq)$ endowed with an involution "$*$" defined by

$$a^* = e, b^* = c, d^* = d;$$

is a ternary ordered $*$-semigroup with order preserving involution $*$.

$aaa=b$	$aab=b$	$aac=d$	$aad=d$	$aae=d$
$aba=b$	$abb=b$	$abc=d$	$abd=d$	$abe=d$
$aca=d$	$acb=d$	$acc=d$	$acd=d$	$ace=d$
$ada=d$	$adb=d$	$adc=d$	$add=d$	$ade=d$
$aea=d$	$aeb=d$	$aec=d$	$aed=d$	$aee=d$
$baa=b$	$bab=b$	$bac=d$	$bad=d$	$bae=d$
$bba=b$	$bbb=b$	$bbc=d$	$bbd=d$	$bbe=d$
$bca=d$	$bcb=d$	$bcc=d$	$bcd=d$	$bce=d$
$bda=d$	$bdb=d$	$bdc=d$	$bdd=d$	$bde=d$
$bea=d$	$beb=d$	$bec=d$	$bed=d$	$bee=d$
$caa=d$	$cab=d$	$cac=d$	$cad=d$	$cae=d$
$cba=d$	$cbb=d$	$cbc=d$	$cbd=d$	$cbe=d$
$cca=d$	$ccb=d$	$ccc=c$	$ccd=d$	$cce=c$
$cda=d$	$cdb=d$	$cdc=d$	$cdd=d$	$cde=d$
$cea=d$	$ceb=d$	$cec=c$	$ced=d$	$cee=c$
$daa=d$	$dab=d$	$dac=d$	$dad=d$	$dae=d$
$dba=d$	$dbb=d$	$dbc=d$	$dbd=d$	$dbe=d$
$dca=d$	$dcb=d$	$dcc=d$	$dcd=d$	$dce=d$
$dda=d$	$ddb=d$	$ddc=d$	$ddd=d$	$dde=d$
$dea=d$	$deb=d$	$dec=d$	$ded=d$	$dee=d$
$eaa=d$	$eab=d$	$eac=d$	$ead=d$	$eae=d$
$eba=d$	$ebb=d$	$ebc=d$	$ebd=d$	$ebe=d$
$eca=d$	$ecb=d$	$ecc=c$	$ecd=d$	$ece=c$
$eda=d$	$edb=d$	$edc=d$	$edd=d$	$ede=d$
$eea=d$	$eeb=d$	$eec=c$	$eed=d$	$eee=c$

2.Characterization of ordered ternary ∗-Semigroups in which all Ideals are (Weakly) Prime

In the sequel S will denote an ordered ternary $*$-Semigroups. Many of the deepest properties of ordered ternary $*$-semigroup depend on ideals. We shall introduce the basic concepts and derive some crucial important properties. Then they will permit us to characterize ordered ternary $*$-semigroup. Let S be an ordered ternary $*$-semigroup. For $H \subset S$, we denote $(H] := \{t \in S \,/\, t \leq h \text{ for some } h \in H\}$. If $H = \{a\}$, we write $(a]$ instead of $(\{a\}]$.

Definition 2.1. A non-empty subset L $(resp. R)$ of S is called a left (resp. right) ideal of S if

1. $SSL \subset L$ $(resp.\ RSS \subset R)$,
2. $a \in L\ (resp.\ R), b \leq a \implies b \in L\ (resp.\ R)$.

I is called an ideal of S if it is both a left and a right ideal of S.

Definition 2.2. A non-empty subset L $(resp.\ R)$ of S is called a strong left (resp. right) ideal of S if

1. $SSL \subset L$ $(resp.\ RSS \subset R)$,
2. $a \in L$ $(resp.\ R)$, $b \leq a \Longrightarrow b \in L$ $(resp.\ R)$,
3. $a \in L$ $(resp.\ R) \Longrightarrow a^* \in L$ $(resp.\ R)$.

Proposition 2.3. Any strong left ideal is a strong right ideal and vice versa. So there is no need to distinguish between strong left and strong right ideal, we simply talk about a strong ideal.

Proof. Let L be a left ideal, $a \in L, s_1, s_2 \in S$, then $a^* \in L$, $s_1^*, s_2^* \in S$ so $s_2^*s_1^*a^* \in L$ so $(s_2^*s_1^*a^*)^* \in L$, but $(s_2^*s_1^*a^*)^* = as_1s_2$ and then $LSS \subset L$. The two other properties hold from the definition of the the left ideal.

Proposition 2.4. The union of strong (resp. left, resp. right) ideals is a strong (resp. left, resp. right) ideal

Proof. Let $(A_i)_{i\in I}$ be a family of strong ideals of S, then :

1. Let $a \in \bigcup_{i\in I} A_i$ and $s_1, s_2 \in S$. $a \in \bigcup_{i\in I} A_i \implies \exists\, i \in I$ such that $a \in A_i$ so, us A_i is a strong ideal, $s_1 s_2 a \in A_i$ and then $s_1 s_2 a \in \bigcup_{i\in I} A_i$.
2. Easy the check that $a \leq b \in \bigcup_{i\in I} A_i \implies a \in \bigcup_{i\in I} A_i$.
3. It is trivial that $a \in \bigcup_{i\in I} A_i \implies a^* \in \bigcup_{i\in I} A_i$.

Proposition 2.5. If I is a strong ideal then $I^* = I$.

Proof. Let I be a strong ideal and $a \in I$ then $a^* \in I$ and therefore $(a^*)^* \in I^*$ but $(a^*)^* = a$ so $I \subset I^*$. The other inclusion can be reached by the same arguments.

Proposition 2.6. Let $*$ be an order preserving involution on S. If I is a left ideal then I^* is a right ideal and vice versa.

Proof.

1.Let $a \in I^*$ and $s_1, s_2 \in S$. There exist $b \in I$ such that $a = b^*$, $as_1s_2 = (s_2^* s_1^* a^*)^* = (s_2^* s_1^* b\,)^*$.

But since $b \in I$ and I is a left ideal then $s_2^* s_1^* b \in I$ and its involute element is in I^* and finally $as_1s_2 \in I^*$.

Let $a \leq b \in I^*$. Then there exit $c \in I$ such that $a \leq b = c^*$ and then $a^* \leq c$ since $*$ is an order preserving involution. But I is a left ideal and $c \in I$ then $a^* \in I$ and therefore $a = (a^*)^* \in I^*$.

Proposition 2.7. For $a \in S$ if $*$ is an order preserving involution then,

$$I(a) = (b \cup SSb \cup bSS \cup SS(\,bSS\,); b \in \{\,a, a^*\}\,]\,,$$

is a strong ideal called the strong ideal generated by a and

$$I_w(a) = (a \cup SSa \cup aSS \cup SS(aSS\,)],$$

is an ideal called the weakly ideal generated by a.

Proof.

a. Let $t \in I(a)$ then there exists $h \in b \cup SSb \cup bSS \cup SS(\,bSS\,)$ such that $t \leq h$. For any $s_1, s_2 \in S\,; s_1s_2t \leq s_1s_2h$ and it is easy to see that the element s_1s_2h is always in $SSb \cup SS(\,bSS\,)$ so $t \in I(a)$.
b. It is easy using the transitivity of $\leq$, to see that if $t \leq h \in I(a)$ then there exist $k \in b \cup SSb \cup bSS \cup SS(\,bSS\,)$ such that $h \leq k$ and then $t \leq k$ and therefore $t \in I(a)$.
c. If $t \in I(a)$ then there exists $h \in b \cup SSb \cup bSS \cup SS(\,bSS\,)$ such that $t \leq h$. So $a^* \leq h^*$ and from the different forms that h can take in the set
d. $b \cup SSb \cup bSS \cup SS(\,bSS\,)$; h^* is also in this set and then $t^* \in I(a)$.
    a. Let $t \in I_w(a)$ then there exists $h \in a \cup SSa \cup aSS \cup SS(\,aSS\,)$ such that $t \leq h$. For any $s_1, s_2 \in S$; $s_1s_2t \leq s_1s_2\, h$ and it is easy to see that the element $s_1s_2\, h$ is always in $SSa \cup SS(\,aSS\,)$ so $t \in I_w(a)$.
    b. It is easy using the transitivity of $\leq$, to see that if $t \leq h \in I_w(a)$ then there exist $k \in a \cup SSa \cup aSS \cup SS(\,aSS\,)$ such that $h \leq k$ and then $t \leq k$ and therefore $t \in I_w(a)$.

Proposition 2.8. Let S be a ternary ordered $*$-semigroup with order preserving involution.

1. $A \subset (A]$ for any $A \subset S$.
2. $(A] \subset (B]$ for any A, B with $A \subset B \subset S$.
3. $(A](B](C] \subset (ABC]$ for any $A, B, C \subset S$.
4. $\big((A]\big] = (A]$ for any $A \subset S$.
5. $(I\,] = I$ for any strong ideal (resp. left ideal, right ideal, ideal) I of S.
6. $(A\,B\,C]$ and $A \cap B$ are ideals for any ideals A, B, C of S.
7. If $ABC = CBA$ then $(ABC]$ is a strong ideal for any strong ideals A, B, C of S.
8. $(SaS]$ is an ideal for any $a \in S$ and if in addition $a^* = s\,s'a$ then $(SaS]$ is a strong ideal.

Proof. The first and the second assertions are easy to prove.

1. Let $a \in (A], b \in (B], c \in (C]$ there exist $h_1 \in A, h_2 \in B, h_3 \in C$ such that $a \leq h_1, b \leq h_2, c \leq h_3$. $abc \leq h_1bc \leq h_1h_2c \leq h_1h_2h_3$ and $h_1h_2h_3 \in ABC$ then $abc \in (ABC]$.
2. From the first assertion the relation is easy to establish.
3. From the first assertion it suffices to prove that $(I] \subset I$. Let $t \in (I]$ then there exist $h \in I$ such that $t \leq h$. Since I is an ideal t is also in I and then $(I] \subset I$.
4. The intersection of ideals is an ideal is simple to check. Let $t \in (ABC]$ then $t \leq abc$ for some $a \in A, b \in B$ and $c \in C$. For $s_1, s_2 \in S$ we have $s_1s_2t \leq s_1s_2\,(abc) = (s_1s_2a\,)bc$. But since A is an ideal then (s_1s_2a) belongs to A and therefore $(s_1s_2a)bc$ is in ABC and s_1s_2t belongs to $(ABC]$. The other properties are simple to get.
5. Using the previous property, to prove the assertion 7; it suffices to prove that $t \in (ABC]$ implies that t^* is in $(ABC]$.
   But $t \in (ABC]$ implies $t \leq abc$ for some $a \in A, b \in B, c \in C$ and then
   $t^* \leq c^*b^*a^* \in CBA$ since A, B, C are strong ideals. From the equality $ABC = CBA$ we deduce that $c^*b^*a^* \in ABC$ and finally $t^* \in (ABC]$.
6. The proof ensues from the property $ss'(s_1as_2) = (ss's_1)as_2$.

Proposition 2.9. Let S be a ternary ordered $*$-semigroup with order preserving involution.

1. $(bSa]^* = (a^*S\,b^*]$ for any $a, b \in S$.
2. $(SaS]^* = (Sa^*S]$ for any $a \in S$.
3. I^* is an ideal for any ideal I of S.

Proof.

1. Let $t \in (bSa]^*$ then there exist $t_1 \in (bSa]$ such that $t = t_1^*$ and therefore $t_1 \leq bsa$ for some $s \in S$. We then get $t = t_1^* \leq (bsa)^* = a^*s^*b^*$ which implies $t \in (a^*Sb^*]$. The other inclusion is trivial.
2. Let $t \in (SaS]^*$ then there exist $t_1 \in (SaS]$ such that $t = t_1^*$ but $t_1 \leq s_1as_2$ for some $s_1, s_2 \in S$.
3. Finally $t = t_1^* \leq s_2^*a^*s_1^* \in Sa^*S$ and then $t \in (Sa^*S]$. Now let $t \in (Sa^*S]$ then $t \leq s_1a^*s_2 \implies t^* \leq s_2^*a\,s_1^* \implies t^* \in (SaS]$ and then $t = (t^*)^* \in (SaS]^*$.
4. Let $t \in I^*$ then there exist $t_1 \in I$ such that $t = t_1^*$, so for $s_1, s_2 \in S; s_1s_2t = s_1s_2t_1^*$. $t_1 \in I, I$ ideal imply $t_1s_1^*s_2^* \in I \implies (t_1s_1^*s_2^*)^* \in I^*$ and therefore $s_1s_2t = s_1s_2t_1^* \in I^*$. Let $t \in S$ such that $t \leq t_1$ with $t_1 \in I^*$. $t_1 \in I^* \implies \exists t_2 \in I$ such that $t_1 = t_2^*$. So $t^* \leq t_1^* = t_2 \in I$ and since I is an ideal then $t^* \in I$ and finally $t \in I^*$

Definition 2.10. Let S be an ordered $*$-semigroup and P be an ideal of S. The ideal P is said to be prime if $ABC \subset P$ implies $A^* \subset P$ or $B^* \subset P$ or $C^* \subset P$. Equivalently: $abc \in P$ implies $a^* \in P$ or $b^* \in P$ or $c^* \in P$.

Definition 2.11. Let S be an ordered $*$-semigroup an element $a \in S$ is said to be prime if for all $s_1, s_2, s_3 \in S; s_1s_2s_3 \in I(a) \implies s_i^* \in I(a)$ for some $1 \leq i \leq 3$; (that is; the ideal generated by a is prime).

Definition 2.12. Let S be an ordered $*$-semigroup and P be an ideal of S. The ideal P is said to be weakly prime if for any ideals
A, B, C the relation $ABC \subset P$ implies $A^* \subset P$ or $B^* \subset P$ or $C^* \subset P$. Equivalently: $abc \in P \Longrightarrow a^* \in P$ or $b^* \in P$ or $c^* \in P$.

Definition 2.13. Let S be an ordered $*$-semigroup and P be an ideal of S. The ideal P is said to be semiprime if for any ideal A the relation $AAA \subset P \Rightarrow A^* \subset P$. Equivalently: $aaa \in P \Longrightarrow a^* \in P$.

Proposition 2.14. Let S be an ordered $*$-semigroup and P be a strong ideal of S. P is weakly prime if and only if it is prime.

Proof. It is clear that if P is prime it is weakly prime.

For the converse; let A, B, C be subsets of S such that $A.B.C \subset P$.

$A.B.C \subset P \Longrightarrow (A.B.C] \subset (P] = P$ since P is a strong ideal [proposition 2.7(5)]; but by the same proposition $(A](B](C] \subset (ABC]$ so $(A](B](C] \subset P$ and since P is weakly prime the $(A] \subset P$ or $(B \subset P$ or $(C] \subset P$ and finally $A \subset (A] \subset P$ or $B \subset (B] \subset P$ or $C \subset (C] \subset P$ and P is prime.

Theorem 2.15. Let S be a ternary ordered $*-$semigroup with order preserving involution $*$. An ideal of S is a strong and prime if and only if it is both weakly prime and semiprime. Furthermore, if S is commutative, then the prime and weakly prime ideals coincide

Proof. Suppose that I is a prime ideal of S. It is trivial that I is both weakly prime and semiprime.

Conversely; let an ideal I be both weakly prime and semiprime. If $a \in I$ then $aaa \in I$ and as I is semiprime then $a^* \in I$ and then I is a strong ideal. By the above proposition I is prime.

Proposition 2.16. Let S be an ordered and principal $*$-semigroup (e.g. its ideals are of the form $I(a)$ with $a \in S$). If $a \in S$ is maximal then $I(a)$ is a maximal ideal.

Proof. Let J be an ideal of S such that $I(a) \subset J$. Since any ideal of S is principal then there exist $b \in S$ such that $J = I(b)$ and then $a \in I(a) \subset I(b)$ imply that $a \leq b$ and by the maximality of a then $b = a$ and finally $I(b) = I(a)$. and $I(a)$ is then maximal.

Proposition 2.17. Let S be an ordered $*-$semigroup with order preserving involution $*$. The following statements are equivalent:

1. $(I^*I^*I^*] = I$ for any ideal I of S,
2. $I^* \cap J^* \cap K^* = (IJK]$ for any ideals I, J, K of S,
3. $I_w(a) \cap I_w(b)] \cap I_w(c) = (I_w(a)^* I_w(b)^* I_w(c)^*]$ (resp. $I(a) \cap I(b) \cap I(c) = (I(a)^* I(b)^* I(c)^*]$) for any $a, b, c \in S;$
4. $I(a) = \left(I(a^*)\right) I(a^*) I(a^*)]$ for
5. any $a \in S$.

Proof.

1. 1) $\Longrightarrow$ 2).

Since I^*, J^* and K^* are ideals, by Propositions 2.8. We then have
$$(IJK] \subset (ISS] \subset (I] = \left((I^*I^*I^*]\right] = (I^*I^*I^*] \subset (I^*] = I^*$$
Similarly
$$(IJK] \subset (SJS] \subset (J] = \left((J^*J^*J^*]\right] = (J^*J^*J^*] \subset (J^*] = J^*$$
and
$$(IJK] \subset (SSK] \subset (K] = \left((K^*K^*K^*]\right] = (K^*K^*K^*] \subset (K^*] = K^*$$
Thus $(IJK] \subset I^* \cap J^* \cap K^*$.

Since $I^* \cap J^* \cap K^*$ is an ideal, we have
$I^* \cap J^* \cap K^* = ((I^* \cap J^* \cap K^*)^*(I^* \cap J^* \cap K^*)^*(I^* \cap J^* \cap K^*)^*] =$
$\left((I \cap J \cap K)\,(I \cap J \cap K)\,(I \cap J \cap K)\,\right] \subset (IJK]$, therefore
$I^* \cap J^* \cap K^* = (IJK]$.

2) $\Longrightarrow$ 3):

Since $I_w(a), I_w(b), I_w(c)$ are ideals so are $\left(I_w(a)\right)^*$, $\left(I_w(b)\right)^*, \left(I_w(c)\right)^*$ and then by taking in 2) $I = \left(I_w(a)\right)^*$, $J = \left(I_w(b)\right)^*, K = \left(I_w(c)\right)^*$, the result fallows.

3) $\Longrightarrow$ 4):

To get this implication it suffices to prove that $I(a^*) = I(b^*)$ and apply 2 to the case where $a = b = c$.

Clearly $a^* \in \left(I(a)\right)^*$ and then $\left(I(a^*)\right)^* \subset \left(I(a)\right)^*$.

Let $x \in \left(I(a)\right)^*$ then $x \in I(a)$, this means that $x^* \leq a$ or $x^* \leq s_1 s_2 a$ or $x^* \leq a s_1 s_2$ or $x^* \leq s_1 s_2 (a s_3 s_3)$ and as the involution is order preserving we get
$x \leq a^*$ or $x \leq s_1' s_2' a^*$ or $x \leq a^* s_1' s_2'$ or $x \leq s_1' s_2' (a^* s_3' s_4')$ and then $x \in I(a^*)$.

Finally $I(a^*) = \left(I(a)\right)^*$.

4) $\Longrightarrow$ 1):

Let $a \in I$ and we have to prove that $a \in (I^*I^*I^*]$.

If $a \in I \Longrightarrow a \leq a \in a \cup SSa \cup aSS \cup SS(aSS)$ and then $a \in I(a)$. From the condition 4) we deduce that $a \in (I(a^*)I(a^*)I(a^*)]$ and then $a \leq b_1 b_2 b_3$ where $b_i \in I(a^*)$. But $b_i \in I(a^*) \Leftrightarrow b_i \in a^* \cup SSa^* \cup a^*SS \cup SS(a^*SS)$. We can then write
$$b_i = \begin{cases} a^* \\ s_1 s_2 a^* \\ a^* s_1 s_2 \\ s_1 s_2 (a^* s s') \end{cases}$$
As $a \in I, a^* \in I^*$ and then $b_i \in I^*$ and finally $a \in (I^*I^*I^*]$.

Now let $a \in (I^*I^*I^*]$ then $a \leq b_1 b_2 b_3$ for some $b_1, b_2, b_3 \in I^*$ so $a \leq b_1^* b_2^* b_3^*$ and as $b_1^* \backslash in\ I$, $b_3^* b_2^* b_1^* \in I$ and then $a \in I^*$.

Theorem 2.18. Let S be an ordered semigroup with order preserving involution $*$. The ideals of S are weakly prime if and only if $I^* = (III]^*$ for any ideal I of S and any two ideals are comparable under the inclusion.

Proof.

1. " $\Longrightarrow$"

Let I, J be two ideals of S. From the proposition 2.8, J^* and $(IJJ^*]$ are ideals and then $(IJJ^*]$ is a weakly prime (that is if $abc \in (IJJ^*]$, there exist $i \in I, j_1, j_2 \in J$ such that $abc \leq i j_1 j_2 \in I$, since I is an ideal and therefore $abc \in I$ which implies $a \in I$ or $b \in I$ or $c \in I$. But $IJJ^* \subset (IJJ^*]$ implies $I^* \subset (IJJ^*]$ or $J^* \subset (IJJ^*]$ or $J \subset (IJJ^*]$.

If $I^* \subset (IJJ^*]$ then $I^* \subset (SSj^*] \subset (J^*]$ and therefore $I \subset J$.

If $J^* \subset (IJJ^*]$ then $J^* \subset (ISS] \subset (I] = I$ therefore $J \subset I$.

If $J \subset (IJJ^*]$ then $J \subset (ISS] \subset (I] = I$ therefore $J \subset I$.

next we have to prove that $I^* = (III]$. Since $III \subset (III]$ and $(III]$ is weakly prime then $I^* \subset (III]$. Now let $x \in (III]$ then $x \leq a_1 a_2 a_3$ for some $a_1, a_2, a_3 \in I$. From the inclusion $I^* \subset (III]$ we get $a_1^* \leq u_1 v_1 w_1 a_2^* \leq u_2 v_2 w_2 a_3^* \leq u_3 v_3 w_3$ for some

$u_i, v_i, w_i \in I$ and then

$$x \le a_1a_2a_3 \le (u_1v_1w_1)^*(u_2v_2w_2)^*(u_3v_3w_3)^* \in (III)^*(III)^*(III)^* = (I^*I^*I^*)(I^*I^*I^*)(I^*I^*I^*) \subset I^*$$

since I^* is an ideal and consequently $x \in (I^*] = I^*$ and therefore $(III] \subset I^*$).

2. " $\Leftarrow$ "

Let I, J, K, L three ideals such that $IJK \subset L$. Since for any ideal $A^* = (AAA]$, by proposition 2.17 ; we have $I^* \cap J^* \cap K^* = (IJK]$ and as any two ideals are comparable among I, J, K there exists one ideal included in the two others, suppose that $I \subset J$ and $I \subset K$, then $I^* \subset J^*$ and $I^* \subset K^*$ so $I^* = I^* \cap J^* \cap K^* \subset (I \cap J \cap K]$ by proposition 2.17. But by hypothesis; $(I \cap J \cap K] \subset (L] = L$ and thus L is weakly prime

Definition 2.19. An ordered $*$ ternary semigroup S is called intra-regular if $a \in (S(a^*a^*a^*)S]$ for any $a \in S$.

Proposition 2.20. Let S be an ordered $*$-semigroup. Then S is intra-regular if and only if the ideals of S are semiprime.

Proof. Suppose that I is an ideal of S with $aaa \in I$ for some $a \in S$. Since S is intra-regular, we have $a^* \in (S(aaa)S] \subset (SIS] \subset (I] = I$ and hence I is semiprime.

Conversely, suppose that a is an element of S. Clearly $(S(aaa)S]$ is an ideal. So $(S(aaa)S]$ is semiprime by hypothesis and then; since $(a^*a^*a^*)(a^*a^*a^*)(a^*a^*a^*) \in S(aaa)S \in (S(aaa)S]$ we have $(a^*a^*a^*) \in (S(aaa)S] \Leftrightarrow aaa \in (S(aaa)S]$ and for the same reason $a^* \in (S(aaa)S]$. But this last relation implies $a^*a^*a^* \in (S(aaa)S]$ and we finally get $a \in (S(aaa)S]$ and S is intra-regular.

Proposition 2.21. Let S be an ordered $*$-semigroup. If S is intra-regular, then $(S\big((xyz)S\big] = \big(SS(x^*z^*y^*)\big)SS)]$ for any $x, y, z \in S$.

Proof. Let $x, y, z \in S$. Since S is intra-regular, we have $xyz \in (S((xyz)^*(xyz)^*(xyz)^*)S] = (S\big((z^*y^*x^*)(z^*y^*x^*)(z^*y^*x^*)\big)S]$. Since

$$s_1\big((z^*y^*x^*)(z^*y^*x^*)(z^*y^*x^*)\big)s_2 = (s_1z^*y^*)(x^*z^*y^*)x^*(z^*y^*x^*))s_2 = s_3(x^*y^*z^*)(x^*(z^*y^*x^*)s_2) = s_3(x^*y^*z^*)s_4$$

where $s_3 = s_1z^*y^*$ and $s_4 = x^*(z^*y^*x^*)s_2$, we can deduce that $(S((xyz)^*(xyz)^*(xyz)^*)S] \subset (S(x^*z^*y^*)S]$ and then $xyz \in (S\,(x^*z^*y^*)S]$.

Then $xyz \le s(x^*z^*y^*)s'$ for some $s, s' \in S$) and then if $s_{1\,,s_2}$ are in S;

$s_1(xyz)s_2 \le s_1(s(x^*z^*y^*)s')s_2 =$
$s_1s\,[(x^*z^*y^*)s's_2 \in SS\big((x^*z^*y^*)SS\big) \subset (SS\big((x^*z^*y^*)SS\big]$ and then

$(S(xyz)S] \subset (SS\big((x^*z^*y^*)SS\big)]$.

For the other inclusion, let

$s_1s_2\big((x^*z^*y^*)ss'\big) = s_1(s_2(x^*z^*y^*)s)s'$.

But by symmetry

$s_2(x^*z^*y^*)s \in ((SS(xzy)SS)]$, so

$s_2(x^*z^*y^*)s \le s_3s_4((xzy)s_5s_6)$ and then

$$s_1(s_2(x^*z^*y^*)s)s' \le s_1\big[s_3s_4\big((xzy)s_5s_6\big)\big]s' ==$$

$= (s_1s_3s_4)(xzy)(s_5s_6s') \in S(xzy)S \subset (S(xzy)S]$ and finally $\big(SS\big((x^*z^*y^*)SS\big)\big] \subset (S(xzy)S]$.

Definition 2.22. A ternary semigroup S is said to be commutative if for all $a, b, c \in S$; $abc = bca = cba = bac = acb = cab$. and circularly commutative if $abc = bca = cab$

Proposition 2.23. Let S be an ordered $*$-semigroup with order preserving involution $*$. If the ideals of S are semiprime, then

1. $I(x) = (SxS]$ for any $x \in S$, and
2. $I(xyz) \subset I(x) \cap I(y) \cap I(z)$ for any $x, y, z \in S$.

Proof.

1. Let x be an element in S. Note that $(SxS]$ is an ideal whence is semiprime. Applying this fact and $(x^3)(x^3)(x^3) = \big((xxx)xx\big)x(xxx) \in (SxS]$ yields $(x^3)^* = x^*x^*x^* \in (SxS]$.
   Applying another time the fact: $(SxS]$ is semiprime, we get $x \in (SxS]$ and then $I(x) \subset (SxS]$. Furthermore $(SxS] \subset (x \cup xS \cup Sx \cup SxS] = I(x)$. Hence $I(x) = (SxS]$
2. Since $xyz \in I(x)SS \subset I(x)$, we have $I(xyz) \subset I(x)$. Also $I(xyz) \subset I(y)$ and $I(xyz) \subset I(z)$ because $xyz \in SI(y)S \subset I(y)$ and $xyz \in SSI(z)S \subset I(z)$. Thus
3. $I(xyz) \subset I(x) \cap I(y) \cap I(z)$.

Proposition 2.24. Let S be a commutative ordered $*$-semigroup with order preserving involution $*$. If the ideals of S are semiprime,
then $I(xyz) \subset I(x) \cap I(y) \cap I(z)$ for any $x, y, z \in S$.

Proof. From proposition 2.23; it remains to prove that the commutativity implies the other inclusion, that is: $I(xyz) \supset I(x) \cap I(y) \cap I(z)$ for any $x, y, z \in S$.

Let $a \in I(x) \cap I(y) \cap I(z)$, then $a \in (SxS] \cap (SyS] \cap (SzS]$by the first statement, whence $a \le u_1xu_2$, $b \le v_1yv_2$ and $c \le w_1zw_2$ for some $u_1, u_2, v_1, v_2, w_1, w_2 \in S$. So

$a^*a^*a^* \le (u_2^*x^*u_1^*)(v_2^*y^*v_1^*) =$
$(w_2^*z^*w_1^*) = (u_2^*u_1^*x^*)(y^*v_2^*v_1^*)(z^*w_2^*w_1^*)$
$= u_2^*u_1^*[x^*(y^*v_2^*v_1^*)(z^*w_2^*w_1^*)] =$
$u_2^*u_1^*\big[\,x^*y^*\big(v_2^*v_1^*(z^*w_2^*w_1^*)\big)\big]$
$= u_2^*u_1^*\big[x^*y^*\big((v_2^*v_1^*z^*)w_2^*w_1^*\big)\big] =$
$u_2^*u_1^*[\,x^*y^*((z^*v_2^*v_1^*)w_2^*w_1^*)] =$
$u_2^*u_1^*\big[x^*y^*\big(z^*v_2^*(v_1^*w_2^*w_1^*)\big)\big] =$
$u_2^*u_1^*[\,(x^*y^*z^*)v_2^*(v_1^*w_2^*w_1^*)] \in SS((xyz)SS)$

and then $a^*a^*a^* \in I(xyz)$ and since $I(xyz)$ is semiprime we get $a \in I(xyz)$.

Theorem 2.25. Let S be an ordered $*$-semigroup with order preserving involution $*$. If the ideals of S are prime; then S is intra-regular and any two ideals are comparable under the inclusion relation.

Proof. If the ideals are prime, then they are weakly prime, and hence Theorem 2.18 implies that any two ideals are comparable. Let $a \in S$. Note that $(S(a^*a^*a^*)S]$ is an ideal by Proposition 2.1, whence is prime. Therefore as

$[(a^*a^*a^*)(a^*a^*a^*)(a^*a^*a^*)]$
$[(a^*a^*a^*)(a^*a^*a^*)(a^*a^*a^*)]$
$[(a^*a^*a^*)(a^*a^*a^*)(a^*a^*a^*)]$
$= (a^*a^*a^*)(a^*a^*a^*)[(a^*a^*a^*)$
$[(a^*a^*a^*)(a^*a^*a^*)(a^*a^*a^*)]$
$[(a^*a^*a^*)(a^*a^*a^*)(a^*a^*a^*)]]$
$\in (S(a^*a^*a^*)S]$, then $[(aaa)(aaa)(aaaa)] \in (S(a^*a^*a^*)S]$ which implies that $(aaa)^* \in (S(a^*a^*a^*)S]$ and finally $a \in (S(a^*a^*a^*)S]$ and S is then intra-regular.

Corollary 2.26. Let S be a commutative ordered $*$-semigroup with order preserving involution $*$. The ideals of S are prime if and only if S is intra-regular and any two ideals are comparable under the inclusion relation.

Proof. From the theorem 2.25 it remains to prove that the commutativity implies the if S
is intra-regular and any two ideals are comparable under the inclusion relation; then the ideals of S are prime.

Let I be an ideal of S and $abc \in I$. We claim that $a^* \in I$ or $b^* \in I$ or $c^* \in I$. By virtue of Proposition 2.22, $I(a)$ is semiprime. Thus
$aaa \in I(a)$ implies $a^* \in I(a)$, $b^* \in I(b)$ and $c^* \in I(c)$ are proved similarly. Furthermore by hypothesis one of the ideals is included in the two others. Suppose $I(a) \subset I(b)$ and $I(a) \subset I(c)$ so
$a^* \in I(a) \subset I(a) \cap I(b) \cap I(c) = I(abc) \subset I$ since $abc \in I$.

Definition 2.27. An element a of a ternary semigroup is said to be an idempotent if $aaa = a$. The set of the idempotents in a subset A of S will be denoted by $E(A)$. Evidently $E(A) \subset A$.

Proposition 2.28. Let S be a ternary $*$-semigroup and I a semiprime ideal.
$\left(E(I)\right)^* \subset I$.

Proof. Let $a \in E(I) \subset I$, since $a = aaa$ and I is semiprime then $a^* \in I$. But $a^*a^*a^* = a^*$.

3. Characterization of Intra-Regular Ordered $*-$Semigroups

In Section 2 we considered ideals. In this section we shall introduce the notion of filters which will be used to establish some congruence. Once some properties are well made it is not difficult to establish the characterization. For convenience we define $a\ \mathcal{R}\ b$ if and only if $I(a) = I(b)$.

Definition 3.1. Let S be a ternary ordered $*$-semigroup. A subsemigroup $\mathcal{F}$ of S is called a filter if

1. for any $a, b, c \in S, abc \in \mathcal{F}$ implies $a^* \in \mathcal{F}, b^* \in \mathcal{F}$ and $c^* \in \mathcal{F}$,
2. for any $a \in \mathcal{F}$, $b \in S$, $c \geq a$ implies $b \in \mathcal{F}$.

Proposition 3.2. Let a be an element of a ternary ordered $*$ -semigroup S. The intersection of all the filters containing a is a least filter of S containing a it will be denoted by $\mathcal{F}(a)$.

Proof. S is a filter of S containing a.
Let $\mathcal{F}(a) = \bigcap_{i\in I} \mathcal{F}_i$, where the intersection is taken over all the filters containing a. Let $x, y, z \in \mathcal{F}(a)$, then $\forall\ i \in I, x, y, z \in \mathcal{F}_i$. Each set $\mathcal{F}_i$ is a subsemigroup, then $xyz \in \mathcal{F}_i$ for any $i \in I$ and then
$xyz \in \bigcap_{i\in I} \mathcal{F}_i = \mathcal{F}(a)$.

Let $x, y, z \in S$ such that $xyz \in \mathcal{F}(a) = \bigcap_{i\in I} \mathcal{F}_i$.As for all $i \in I, xyz \in \mathcal{F}_i$ and $\mathcal{F}_i$ is a filter then for all $i \in I, x^* \in \mathcal{F}_i, y^* \in \mathcal{F}_i$ and $z^* \in \mathcal{F}_i$ which imply
$x^* \in \bigcap_{i\in I} \mathcal{F}_i = \mathcal{F}(a), y^* \in \bigcap_{i\in I} \mathcal{F}_i = \mathcal{F}(a)$ and $z^* \in \bigcap_{i\in I} \mathcal{F}_i = \mathcal{F}(a)$.

Now let $x \in \mathcal{F}(a) = \bigcap_{i\in I} \mathcal{F}_i$, $y \in S$ such that $y \geq x$.Since for all $i \in I, x \in \mathcal{F}_i$ and $\mathcal{F}_i$ is a filter.
Then for all $i \in I, y \in \mathcal{F}_i$ and then $y \in \bigcap_{i\in I} \mathcal{F}_i = \mathcal{F}(a)$.

Proposition 3.3. Let S be a ternary ordered $*$-semigroup. The relation $\Re$ on S defined by $a\ \Re\ b \iff \mathcal{F}(a) = \mathcal{F}(b)$, is a congruence on S (i.e. an equivalence relation closed for both the ternary operation and the involution).

Proof. It is easy to see that $\Re$ is an equivalence relation. Let $a, b, c, d \in S$ such that $a\Re b$. Since $acd \in \mathcal{F}(acd)$ then $a^* \in \mathcal{F}(acd), c^* \in \mathcal{F}(acd)$
and $d^* \in \mathcal{F}(acd)$ and as $\mathcal{F}(acd)$ is a subsemigrou $a^*a^*a^* \in \mathcal{F}(acd)$,
$c^*c^*c^* \in \mathcal{F}(acd)$ and $d^*d^*d^* \in \mathcal{F}(acd)$ and we finally get $a, c, d \in \mathcal{F}(acd)$.

On the other hand the relations,
$b \in \mathcal{F}(b) = \mathcal{F}(a) \subset \mathcal{F}(acd)$ imply
$b \in \mathcal{F}(acd)$.and then $bcd \in \mathcal{F}(acd)$ and
$$\mathcal{F}(bcd) \subset \mathcal{F}(acd) \qquad (1).$$
Similarly $bcd \in \mathcal{F}(bcd)$ implies $b, c, d \in \mathcal{F}(bcd)$. And as $a \in \mathcal{F}(a) = \mathcal{F}(b) \subset \mathcal{F}(bcd)$, we get $a \in \mathcal{F}(bcd)$ and then $acd \in \mathcal{F}(bcd)$, so $a \in \mathcal{F}(bcd)$
and then $acd \in \mathcal{F}(bcd)$ and
$$\mathcal{F}(acd) \subset \mathcal{F}(bcd) \quad (2)$$
The relations (1) and (2) imply
$$\mathcal{F}(acd) = \mathcal{F}(bcd)$$
Similarly, we use the same techniques to get $\mathcal{F}(cda) = \mathcal{F}(cdb)$ and $\mathcal{F}(cad) = \mathcal{F}(cbd)$.
$a \in \mathcal{F}(a) \Longrightarrow aaa \in \mathcal{F}(a)$, because $\mathcal{F}(a)$ is a subsemigroup and then $a^* \in \mathcal{F}(a)$, ($\mathcal{F}(a)$
is a filter), we then have $\mathcal{F}(a^*) \subset \mathcal{F}(a)$. By symmetry, we obtain
$$a\Re b \iff \mathcal{F}(a) = \mathcal{F}(b) \iff \mathcal{F}(a^*) = \mathcal{F}(b^*) \iff a^*\Re b^*$$

Definition 3.4. A congruence $\Re$ on an ordered $*-$semigroup S is called semilattice
congruence if $a^*a^*a^*\Re\ a$ and $a_1a_2a_3\ \Re\ a_{\sigma(1)}a_{\sigma(2)}a_{\sigma(3)}$, for any $a, a_i \in S$ and $\sigma \in S_3$. A semilattice congruence $\Re$ on S is called complete if $a \leq b$ implies $a\ \Re\ aab$
and $a\ \Re\ abb$.

Proposition 3.5. Let S be an ordered semigroup. Then the relation R of the proposition 3.3
is a complete semilattice congruence on S.

Proof.

1. Let $a \in S$, since $a \in \mathcal{F}(a)$ and $\mathcal{F}(a)$ is subsemigroup then $aaa \in \mathcal{F}(a)$ which implies that $a^* \in \mathcal{F}(a)$ and therefore $a^*a^*a^* \in \mathcal{F}(a)$. Consequently, $\mathcal{F}(a^*a^*a^*) \subset \mathcal{F}(a)$. On the other hand $a^*a^*a^* \in \mathcal{F}(a^*a^*a^*)$ so $a \in \mathcal{F}(a)$ and then $\mathcal{F}(a) \subset \mathcal{F}(a^*a^*a^*) \subset \mathcal{F}(a)$ and finally
$$\mathcal{F}(a) = \mathcal{F}(a^*a^*a^*) \iff (aa^*a^*)\ \Re\ a.$$

2. Since $a_1a_2a_3 \in \mathcal{F}(a_1a_2a_3)$, then $a_1^*, a_2^*, a_3^* \in \mathcal{F}(a_1a_2a_3)$ and therefore $a_1^*\,a_1^*a_1^*,\ a_2^*a_2^*a_2^*,\ a_3^*a_3^*a_3^* \in \mathcal{F}(a_1a_2a_3)$ which imply that $a_1, a_2, a_3 \in \mathcal{F}(a_1a_2a_3)$ and then $a_{\sigma(1)}a_{\sigma(2)}a_{\sigma(3)} \in \mathcal{F}(a_1a_2a_3)$ and $\mathcal{F}(a_{\sigma(1)}a_{\sigma(2)}a_{\sigma(3)}) \subset \mathcal{F}(a_1a_2a_3)$.

Similarly we can prove that
$$\forall\sigma \in S_3,\ \mathcal{F}(a_1a_2a_3) \subset \mathcal{F}(a_{\sigma(1)}a_{\sigma(2)}a_{\sigma(3)}).$$

3. Now let $a, b \in S$ such that $a \leq b$. In one side, $aab \in \mathcal{F}(aab)$ implies $a^* \in \mathcal{F}(aab)$ and $b^* \in \mathcal{F}(aab)$.
Then $a^*a^*a^* \in \mathcal{F}(aab)$ which implies $a \in \mathcal{F}(aab)$. Finally
$$\mathcal{F}(a) \subset \mathcal{F}(aab). \qquad (3)$$

On the other hand since $a \leq b$ and $a \in \mathcal{F}(a)$ we get $b \in \mathcal{F}(a)$ and then
$aab \in \mathcal{F}(a)$, so $\mathcal{F}(aab) \subset \mathcal{F}(a)$ (4)
The relations (3) and (4) imply
$$\mathcal{F}(aab) = \mathcal{F}(a) \iff a\ \Re\ (aab).$$

Similar arguments, we can get

$$\mathcal{F}(aab) = \mathcal{F}(a) \Leftrightarrow a\, \mathfrak{R}\, (abb).$$

Proposition 3.6. Let S be an ordered $*$-semigroup with order preserving involution. If $\mathcal{F}(x) = \{y \in S \mid x \in (Sy^*S]\}$ for any $x \in S$, then S is intra-regular.

Proof. If $\mathcal{F}(x) = \{y \in S \mid x \in (Sy^*S]\}$ for any $x \in S$, since $\mathcal{F}(x)$is a subsemigroup, $x \in \mathcal{F}(x)$ then $xxx \in \mathcal{F}(x)$ and so $x \in (S(xxx)^*S] = (S(x^*x^*x^*)S]$, which means that S is intra-regular.

Theorem 3.7. Let S be an ordered $*$ -semigroup with order preserving involution. If $I = \mathfrak{R}$ then S is intra-regular.

Proof. Recall that

$a\, \mathfrak{R}\, b \Leftrightarrow \mathcal{F}(a) = \mathcal{F}(b)$ and

$a\, I\, b \Leftrightarrow I_w(a) = I_w(b)$.

By the Definition 3.4 and the Proposition 3.5, for any $a \in S, (a, a^*a^*a^*) \in \mathfrak{R}$. As $I = \mathfrak{R}$, $(a, a^*a^*a^*) \in I$. But $a \in I_w(a) = I_w(a^*a^*a^*)$ which implies $a \in I_w(a^*a^*a^*)$ and then

$$a \leq \begin{cases} a^*a^*a^* & (1) \\ (a^*a^*a^*)s_1s_2 \quad s_1s_2 \in S & (2) \\ s_1s_2(a^*a^*a^*) \quad s_{1,}s_2 \in S & (3) \\ s_1s_2[(a^*a^*a^*)\, ss'] \quad s_{1,}s_2, s, s' \in S & (4) \end{cases}$$

Case (a).

From $a \leq a^*a^*a^*$ we get $a^* \leq aaa$ and then $a \leq a^*a^*a^* \leq (aaa)a^*a^* \leq \left[\left(aa(a^*a^*a^*)\right)\right]a^*a^* = [(aaa^*)a^*a^*]a^*a^* = (aaa^*)(a^*a^*a^*)a^* \in (S(a^*a^*a^*)S]$. Then $a \in (S(a^*a^*a^*)S]$.

Case (b).

$a \leq (a^*a^*a^*)s_1s_2 \leq [(aaa)a^*a^*]s_1s_2 \leq \left[\left(aa(aa^*a^*)\right)\right]s_1s_2 \leq \left[aa[(a^*a^*a^*)a^*a^*]\right]s_1s_2 = [(aaa^*)(a^*a^*a^*)a^*]s_1s_2 = (aaa^*)(a^*a^*a^*)(a^*s_1s_2) \in (S(a^*a^*a^*)S]$ and then $a \in (S(a^*a^*a^*)S]$.

For the cases (c) and (d), we use similar arguments and techniques to prove that $a \in (S(a^*a^*a^*)S]$. Finally S is intra-regular stated.

Example 3.8.

As it is easy to see that $S = (\mathbb{Z}^-, \times, \leq)$ where $\mathbb{Z}^- = \{\dots, -4, -3, -2, -1\}$ (the set of negative integers) is a ternary semigroup. If we define the involution to be the identity mapping and since $\forall b, c \in S$, $bc \geq 1$, then for all $a \in S, a \geq abc$ and consequently $a \in (S(aaa)S] = (S(a^*a^*a^*)S]$ and S is intra-regular. We also have for all $a \in S, (SaS] = \{x \in S \mid x \leq a\}$. But $x \leq a$ implies that $x \leq (-1)a(-1) \in SaS$ and if $b, c \in S, bc \geq 1$ and then $bac \leq a$ (a is negative), from which we get

$$bac \in \{x \in S \mid x \leq a\}.$$

Many other investigations can be done; as studying filters, ideals,...

Remark 3.9. In next work we will study the semigroup $S = (\mathbb{Z}^-, \times, \geq)$, where $*$ is supposed to be a semi-involution (i.e. it verifies only $(a^*)^* = a$. The unary operation $*$ will be defined by

$$a^* = \begin{cases} a+1 & if\ a\ is\ even \\ a-1 & if\ a\ is\ odd. \end{cases}$$

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