APPLICATIONS AND DERIVATION OF OSTROWSKI TYPE INEQUALITIES FOR n-TIMES DIFFERENTIABLE FUNCTIONS FOR SOME EFFICIENT QUADRATURE RULES

Abdul Rehman Kashif[1], Muhammad Shoaib[2] Muhammad Amer Latif[3]

[1]Department of Mathematics, Capital University of Sciences and Technology, Islamabad, Pakistan

[2]Abu Dhabi Men's College, Higher Colleges of Technology, P.O. Box 25035, Abu Dhabi, United Arab Emirates

[3]School of Computational and Applied Mathematics, University of the Witwatersrand, Private Bag 3, Wits 2050, Johannesburg, South Africa

Corresponding Author: kashmology@gmail.com

ABSTRACT: *In this paper we obtain new Ostrowski type inequalities for n-times differentiable functions to derive new and efficient quadrature rules. The error bounds of the quadrature rules are shown to depend on the upper and lower bound of the integrand and its derivatives. The efficiency of the quadrature rules is demonstrated with the help of several examples.*

Keywords: Ostrowski inequality, quadrature rule, absolutely continuous function, Lebesgue p-norm

INTRODUCTION

In 1938, Ostrowski [24] proved a very important inequality, which states that for a function $\gamma: [a_1, a_2] \to \mathbb{R}$ of bounded derivative, the following inequality holds:

$$\left|\gamma(s) - \frac{1}{a_2 - a_1}\int_{a_1}^{a_2} \gamma(s)ds\right| \leq \left(\frac{(s-a_1)^2+(a_2-s)^2}{2(a_2-a_1)}\right) \left\|\gamma'\right\|_\infty$$

for all $s \in [a_1, a_2]$, where $\left\|\gamma'\right\|_\infty = \sup_{s\in[a_1,a_2]} \left|\gamma'(s)\right| < \infty$. (1)

The following result is the extension of the result (1) given by Dragomir and Wang [5, 6] for absolutely continuous functions such that γ' belongs to $L_p[a_1, a_2]$, $1 \leq p < \infty$.

Theorem 1 *Let $\gamma: [a_1, a_2] \to \mathbb{R}$ be absolutely continuous on $[a_1, a_2]$. Then for all $s \in [a_1, a_2]$ and $\frac{1}{p} + \frac{1}{q} = 1$, where p, q are real numbers greater than 1, we have*

$$\left|\gamma(s) - \frac{1}{a_2 - a_1}\int_{a_1}^{a_2} \gamma(\theta)d\theta\right| \leq \begin{cases} \frac{1}{(p+1)^{\frac{1}{p}}}\left(\left(\frac{s-a_1}{a_2-a_1}\right)^{p+1} + \left(\frac{a_2-s}{a_2-a_1}\right)^{p+1}\right) \\ \times (a_2-a_1)^{\frac{1}{q}}\left\|\gamma'\right\|_p \\ \frac{1}{a_2-a_1}\left[\frac{a_2-a_1}{2} + \left|s - \frac{a_1+a_2}{2}\right|\right]\left\|\gamma'\right\|_1. \end{cases} \quad (2)$$

Most recently, Masjed-Jamei and Dragomir [21] have presented some analogues of the Ostrowski's inequality by using the following identities and improved all the results involving Lebesgue p-norms of $\gamma'(s)$, $1 \leq p < \infty$:

$$\int_{a_1}^{a_2} S(s,\theta)\gamma'(\theta)d\theta = (a_2 - a_1)\gamma(s) - \int_{a_1}^{a_2} \gamma(\theta)d\theta, \quad (3)$$

and

$$\int_{a_1}^{a_2} |S(s,\theta)|dt = \frac{1}{2}[(s-a_1)^2 + (a_2-s)^2],$$

where

$$S(s,\theta) = \begin{cases} \theta - a_1, & \text{if } a_1 \leq \theta \leq s \\ \theta - a_2, & \text{if } s < \theta \leq a_2. \end{cases}$$

Moreover, the results given in [21] have advantage over the previous results since the necessary computations to find bounds in these results depend on pre-assigned functions other than γ or γ'. They give error bounds of the midpoint rule and other nonstandard quadrature rules.

The Ostrowski's inequality (1) has been generalized, extended and refined in different ways. Alomari [1], Cerone [2] and Dragomir [11] established Ostrowski type inequalities for the Riemann-Stieltjes integrals. Dragomir [8] proved some Ostrowski type inequalities for Lipschitzian mappings and for monotonic functions. Fink gave Ostrowski type inequalities for functions of bounded variation. Various other Ostrowski type inequalities in one variable and several variables and their applications to numerical analysis and statistics can be found in [4,7], [9-10], [13-22], [26] and in the references of these articles.

A more general form of Ostrowski's result for mappings that posseses nth derivative was given by Milovanovic′ and Pečaric′ in [23, p. 468.] as follows.

Theorem 2 *[23] Let $\gamma: [a_1, a_2] \to \mathbb{R}$ be a mpping which possesses n th derivative with $\left\|\gamma^{(n)}\right\|_\infty := \sup_{\theta\in(a_1,a_2)} \left|\gamma^{(n)}(\theta)\right| < \infty$, $n \geq 1$. Then*

$$\left|\frac{1}{n}\left(\gamma(s) + \sum_{k=1}^{n-1} \frac{n-k}{k!} \cdot \frac{\gamma(a_1)(s-a_1)^k - \gamma^{(k-1)}(a_2)(s-a_2)^k}{a_2-a_1}\right) - \frac{1}{a_2-a_1}\int_{a_1}^{a_2} \gamma(t)dt\right| \leq \frac{\left\|\gamma^{(n)}\right\|_\infty}{n(n+1)!} \cdot \frac{(s-a_1)^{n+1} + (a_2-s)^{n+1}}{a_2-a_1} \quad (4)$$

for all $s \in [a_1, a_2]$.

Cerone et. al [3] apprached to another generality of the Ostrowski inequality for mappings which posesses nth derivatives as mentioned in the following theorem.

Theorem 3 *[3] Let $\gamma: [a_1, a_2] \to \mathbb{R}$ be a mapping with absolute continuity of $\gamma^{(n-1)}$ on $[a_1, a_2]$ such that $\gamma^{(n)} \in L_\infty[a_1, a_2]$. Then for all $s \in [a_1, a_2]$, we have the inequality:*

$$\left|\int_{a_1}^{a_2}\gamma(\theta)d\theta - \sum_{k=0}^{n-1}\left[\frac{(a_2-s)^{k+1}+(-1)^{k+1}(s-a_1)^{k+1}}{(k+1)!}\right]\gamma^{(k)}(s)\right| \le \frac{\|\gamma^{(n)}\|_\infty}{(n+1)!}[(s-a_1)^{n+1}+(a_2-s)^{n+1}] \le \frac{\|\gamma^{(n)}\|_\infty (a_2-a_1)^{n+1}}{(n+1)!}, \quad (5)$$

where

$$\|\gamma^{(n)}\|_\infty := \sup_{t\in[a_1,a_2]}|\gamma^{(n)}(\theta)| < \infty.$$

They used the following Lemma to prove the above result.

Lemma 1 *[3] Let $\gamma:[a_1,a_2]\to\mathbb{R}$ be a mapping with the absolute continuity of $\gamma^{(n-1)}$ on $[a_1,a_2]$. Then for all $s\in[a_1,a_2]$, the following identity holds:*

$$\int_{a_1}^{a_2} S_n(s,\theta)\gamma^{(n)}(\theta)d\theta = (-1)^{n+1}\left(\sum_{k=0}^{n-1}\left[\frac{(a_2-s)^{k+1}+(-1)^k(s-a_1)^{k+1}}{(k+1)!}\right]\times\gamma^{(k)}(s) - \int_{a_1}^{a_2}\gamma(\theta)dt\right), \quad (6)$$

where the kernel $S_n:[a_1,a_2]\to\mathbb{R}$ is given by

$$S_n(s,\theta) = \begin{cases} \dfrac{(\theta-a_1)^n}{n!}, & \text{if } a_1\le\theta\le s, \\ \dfrac{(\theta-a_2)^n}{n!}, & \text{if } s<\theta\le a_2, \end{cases} \quad (7)$$

where $n\ge 1$ is a natural number.

In section 2, we introduce a new analogue of the Ostrowski inequalities for n-times differentiable functions which not only improve the results involving Lebesgue norms of the nth derivative but also contain the results from [21] for $n=1$ as special case. In section 3, we use the inequalities obtained in section 2 to derive new quadrature rules. Their efficiency is demonstrated using specific examples as well as by deriving their respective error bounds.

2 Derivation of Ostrowski type Inequalities for n-times differentiable functions

Throughout in this section we will consider the following notations

$$\rho_n(s;a_1,a_2) := \sum_{k=0}^{n-1}\left[\frac{(a_2-s)^{k+1}+(-1)^k(s-a_1)^{k+1}}{(k+1)!}\right]\gamma^{(k)}(s),$$

$$\tau_n(s;a_1,a_2) := \max\left\{\frac{(s-a_1)^n}{n!},\frac{(a_2-s)^n}{n!}\right\}$$

and

$$\sigma_n := \frac{\gamma^{(n-1)}(a_2)-\gamma^{(n-1)}(a_1)}{a_2-a_1}.$$

Theorem 4 *Let $\gamma:[a_1,a_2]\to\mathbb{R}$ be a mapping with absolute continuity of $\gamma^{(n-1)}$ on $[a_1,a_2]$. If $\eta(s)\le\gamma^{(n)}(s)\le\xi(s)$ for any α and $\xi\in C[a_1,a_2]$ and $s\in[a_1,a_2]$, then the following inequality holds:*

$$\frac{1}{n!}\int_{a_1}^{s}(\theta-a_1)^n\eta(\theta)d\theta + \frac{1}{2n!}\int_s^{a_2}(s-a_2)^n[(\eta(\theta)+\xi(\theta)) + (-1)^{n+1}(\xi(\theta)-\eta(\theta)]d\theta \le (-1)^{n+1}\left(\rho_n(s;a_1,a_2)-\int_{a_1}^{a_2}\gamma(\theta)d\theta\right) \le \frac{1}{n!}\int_{a_1}^{s}(\theta-a_1)^n\xi(\theta)d\theta + \frac{1}{2n!}\int_s^{a_2}(\theta-a_2)^n[(\eta(\theta)+\xi(\theta)) + (-1)^n\big((\xi(\theta)-\eta(\theta)\big)d\theta. \quad (8)$$

Proof: From the identity (6) and the kernel defined by (7), we have

$$\int_{a_1}^{a_2} S_n(s;\theta)\left(\gamma^{(n)}(\theta)\frac{\eta(\theta)+\xi(\theta)}{2}\right)d\theta = \int_{a_1}^{a_2}S_n(s;\theta)\gamma^{(n)}(\theta)dt - \frac{1}{2}\int_{a_1}^{a_2}S_n(s;\theta)(\eta(\theta)+\xi(\theta))d\theta = (-1)^{n+1}\left(\rho_n(s;a_1,a_2)-\int_{a_1}^{a_2}\gamma(\theta)d\theta\right) - \frac{1}{2n!}\left(\int_{a_1}^{s}(s-a_1)^n(\eta(\theta)+\xi(\theta))d\theta + \int_s^{a_2}(\theta-a_2)^n(\eta(\theta)+\xi(\theta))d\theta\right). \quad (9)$$

The assumption $\eta(s)\le\gamma^{(n)}(s)\le\xi(s)$ implies that

$$\left|\gamma^{(n)}(\theta)-\frac{\eta(\theta)+\xi(\theta)}{2}\right| \le \frac{\xi(\theta)-\eta(\theta)}{2} \quad (10)$$

Hence from (9) and (10), we have

$$\left|(-1)^{n+1}\left(\rho_n(s;a_1,a_2)-\int_{a_1}^{a_2}\gamma(\theta)d\theta\right) - \frac{1}{2n!}\left(\int_{a_1}^{s}(\theta-a_1)^n(\eta(\theta)+\xi(\theta))d\theta + \int_s^{a_2}(\theta-a_2)^n(\eta(\theta)+\xi(\theta))d\theta\right)\right|$$

$$\le \int_{a_1}^{a_2}|S_n(s;\theta)|\left|\gamma^{(n)}(\theta)-\frac{\eta(\theta)+\xi(\theta)}{2}\right|d\theta \le \int_{a_1}^{a_2}|S_n(s;\theta)|\left(\frac{\xi(\theta)-\eta(\theta)}{2}\right)d\theta = \frac{1}{2n!}\left(\int_{a_1}^{s}(\theta-a_1)^n(\xi(\theta)-\eta(\theta))d\theta + \int_s^{a_2}(a_2-\theta)^n(\xi(\theta)-\eta(\theta))d\theta\right). \quad (11)$$

By re-arranging (11), the main inequality (8) can be derived.

Corollary 1 *Suppose $\gamma^{(n)}(s)$ is bounded by $\eta(\theta)=\eta_n\theta^n+\eta_{n-1}\theta^{n-1}+\cdots+\eta_0\neq 0$ and $\xi(\theta)=\xi_n\theta^n+$*

$\xi_{n-1}\theta^{n-1}+\cdots+\xi_0\neq 0$. *In this case the main inequality (8) takes the form*

$$\frac{1}{n!}\int_{a_1}^{s}(\theta-a_1)^n(\eta_n\theta^n+\eta_{n-1}\theta^{n-1}+\cdots+\eta_0)d\theta$$
$$+\frac{1}{2n!}\int_{s}^{a_2}(\theta-a_2)^n\big[\big((\eta_n+\xi_n)\theta^n+(\eta_{n-1}+\xi_{n-1})\theta^{n-1}+\cdots+(\eta_0+\xi_0)\big)$$
$$+(-1)^{n+1}\big((\xi_n-\eta_n)\theta^n+(\xi_{n-1}-\eta_{n-1})\theta^{n-1}+\cdots+(\xi_0-\eta_0)\big)\big]d\theta$$
$$\leq(-1)^{n+1}\left(\rho_n(x;a_1,a_2)-\int_{a_1}^{a_2}\gamma(\theta)d\theta\right)$$
$$\leq\frac{1}{n!}\int_{a_1}^{s}(\theta-a_1)^n(\xi_n\theta^n+\xi_{n-1}\theta^{n-1}+\cdots+\xi_0)d\theta$$
$$+\frac{1}{2n!}\int_{s}^{a_2}(\theta-a_2)^n\big[\big((\eta_n+\xi_n)\theta^n+(\eta_{n-1}+\xi_{n-1})\theta^{n-1}+\cdots+(\eta_0+\xi_0)\big)$$
$$+(-1)^{n}\big((\xi_n-\eta_n)\theta+(\xi_{n-1}-\eta_{n-1})\theta^{n-1}+\cdots+(\xi_0-\eta_0)\big)\big]d\theta. \quad (12)$$

Theorem 5 *Let $\gamma:[a_1,a_2]\to\mathbb{R}$ be a mapping with absolute continuity of $\gamma^{(n-1)}$ on $[a_1,a_2]$. If $\eta(s)\leq\gamma^{(n)}(s)$ for any $\eta\in C[a_1,a_2]$ and $s\in[a_1,a_2]$, then the following inequality holds:*

$$\frac{1}{n!}\left(\int_{a_1}^{s}(\theta-a_1)^n\eta(\theta)d\theta+\int_{s}^{a_2}(\theta-a_2)^n\eta(\theta)d\theta\right)-\tau_n(s;a_1,a_2)\times\left((a_2-a_1)\sigma_n-\int_{a_1}^{a_2}\eta(\theta)d\theta\right)$$
$$\leq(-1)^{n+1}\left(\rho_n(s;a_1,a_2)-\int_{a_1}^{a_2}\gamma(\theta)d\theta\right)$$
$$\leq\frac{1}{n!}\left(\int_{a_1}^{s}(\theta-a_1)^n\eta(\theta)d\theta+\int_{s}^{a_2}(\theta-a_2)^n\eta(\theta)d\theta\right)$$
$$+\tau_n(s;a_1,a_2)\left((a_2-a_1)\sigma_n-\int_{a_1}^{a_2}\eta(\theta)d\theta\right)\quad (13)$$

Proof: Since

$$\int_{a_1}^{a_2}S_n(s;\theta)\big(\gamma^{(n)}(\theta)-\eta(\theta)\big)d\theta=(-1)^{n+1}\left(\rho_n(s;a_1,a_2)-\int_{a_1}^{a_2}\gamma(\theta)d\theta\right)-\frac{1}{n!}\left(\int_{a_1}^{s}(\theta-a_1)^n\eta(\theta)d\theta+\int_{s}^{a_2}(\theta-a_2)^n\eta(\theta)d\theta\right)$$

.

Hence we have

$$\left|(-1)^{n+1}\left(\rho_n(s;a_1,a_2)-\int_{a_1}^{a_2}\gamma(\theta)d\theta\right)-\frac{1}{n!}\left(\int_{a_1}^{s}(\theta-a_1)^n\eta(\theta)d\theta+\int_{s}^{a_2}(\theta-a_2)^n\eta(\theta)d\theta\right)\right|$$
$$\leq\int_{a_1}^{a_2}|S_n(s;\theta)|\big(\gamma^{(n)}(\theta)-\eta(\theta)\big)d\theta$$
$$\leq\max_{\theta\in[a_1,a_2]}|S_n(s;\theta)|\int_{a_1}^{a_2}\big(\gamma^{(n)}(\theta)-\eta(\theta)\big)d\theta$$
$$=\tau_n(s;a_1,a_2)\left((a_2-a_1)\sigma_n-\int_{a_1}^{a_2}\eta(\theta)d\theta\right).\quad(14)$$

From (14) one can easily derive (13).

Corollary 2 *If $\eta(\theta)=\eta_0\neq 0$ then equation (13) implies*

$$\left(\frac{\eta_0}{(a_2-a_1)(n+1)!}\right)\big((s-a_1)^{n+1}-(s-a_2)^{n+1}\big)-\tau_n(s;a_1,a_2)(\sigma_n-\alpha_0)$$
$$\leq(-1)^{n+1}\left(\rho_n(s;a_1,a_2)-\int_{a_1}^{a_2}\gamma(\theta)d\theta\right)\leq$$
$$\left(\frac{\eta_0}{(a_2-a_1)(n+1)!}\right)\big((s-a_1)^{n+1}-(s-a_2)^{n+1}\big)+\tau_n(s;a_1,a_2)(\sigma_n-\alpha_0).\quad(15)$$

Theorem 6 *Let $\gamma:[a_1,a_2]\to\mathbb{R}$ be a mapping with absolute continuity of $\gamma^{(n-1)}$ on $[a_1,a_2]$. If $\gamma^{(n)}(s)\leq\xi(s)$ for any $\xi\in C[a_1,a_2]$ and $s\in[a_1,a_2]$, then the following inequality holds:*

$$\frac{1}{n!}\left(\int_{a_1}^{s}(\theta-a_1)^n\xi(\theta)d\theta+\int_{s}^{a_2}(\theta-a_2)^n\xi(\theta)d\theta\right)-\tau_n(s;a_1,a_2)\times\left(\int_{a_1}^{a_2}\xi(\theta)d\theta-(a_2-a_1)\sigma_n\right)$$
$$\leq\frac{(-1)^{n+1}}{a_2-a_1}\left(\rho_n(s;a_1,a_2)-\int_{a_1}^{a_2}\gamma(\theta)d\theta\right)$$
$$\leq\frac{1}{n!}\left(\int_{a_1}^{s}(\theta-a_1)^n\xi(\theta)d\theta+\int_{s}^{a_2}(\theta-a_2)^n\xi(\theta)d\theta\right)+\tau_n(s;a_1,a_2)\times\left(\int_{a_1}^{a_2}\xi(\theta)d\theta-(a_2-a_1)\sigma_n\right).\quad(16)$$

Proof: We observe that

$$\int_{a_1}^{a_2}S_n(s;\theta)\left(\gamma^{(n)}(\theta)-\xi(\theta)\right)d\theta=(-1)^{n+1}\left(\rho_n(s;a_1,a_2)-\int_{a_1}^{a_2}\gamma(\theta)d\theta\right)-\frac{1}{n!}\left(\int_{a_1}^{s}(\theta-a_1)^n\xi(\theta)dt+\int_{s}^{a_2}(\theta-a_2)^n\xi(\theta)d\theta\right).$$

Therefore

$$\left|(-1)^{n+1}\left(\rho_n(s;a_1,a_2)-\int_{a_1}^{a_2}\gamma(\theta)d\theta\right)-\frac{1}{n!}\left(\int_{a_1}^{s}(\theta-a_1)^n\xi(\theta)dt+\int_{s}^{a_2}(\theta-a_2)^n\xi(\theta)d\theta\right)\right|$$
$$\leq\int_{a_1}^{a_2}|S_n(s;\theta)|\left(\left(\xi(\theta)-\gamma^{(n)}(\theta)\right)\right)d\theta$$
$$\leq\max_{t\in[a_1,a_2]}|S_n(s;\theta)|\int_{a_1}^{a_2}\left(\left(\xi(\theta)-\gamma^{(n)}(\theta)\right)\right)d\theta$$

$$= \tau_n(s;a_1,a_2) \times \left(\int_{a_1}^{a_2} \xi(\theta)d\theta - (a_2-a_1)\sigma_n\right). \qquad (17)$$

From (17) one can easily derive (16).

Corollary 3 *If $\xi(\theta) = \xi_0 \neq 0$, then inequality (16) reduces to*

$$\left(\frac{\xi_0}{(a_2-a_1)(n+1)!}\right)((s-a_1)^{n+1} - (s-a_2)^{n+1}) - \tau_n(s;a_1,a_2)(\xi_0-\sigma_n)$$
$$\leq \frac{(-1)^{n+1}}{a_2-a_1}\left(\rho_n(s;a_1,a_2) - \int_{a_1}^{a_2}\gamma(\theta)d\theta\right)$$
$$\leq \left(\frac{\xi_0}{(a_2-a_1)(n+1)!}\right)((s-a_1)^{n+1} - (s-a_2)^{n+1}) + \tau_n(s;a_1,a_2)(\xi_0-\sigma_n). \qquad (18)$$

Derivation of Numerical Quadrature Rules

In this section, we propose some new error bounds for new quadrature rules involving higher order derivatives of the function γ . These error bounds depend on the continuous functions α and ξ which are the upper and lower bounds of the nth derivative of the function γ. In fact, the following new quadrature rules can be obtained while investigating the error bounds using theorems 4, 5 and 6:

$$I_{n,1}(\gamma) := \int_{a_1}^{a_2}\gamma(s)ds \cong \sum_{k=0}^{n-1}\frac{[1+(-1)^k](a_2-a_1)^{k+1}}{2^{k+1}(k+1)!}\gamma^{(k)}\left(\frac{a_1+a_2}{2}\right),$$

$$I_{n,2}(\gamma) := \int_{a_1}^{a_2}\gamma(s)ds \cong \sum_{k=0}^{n-1}\frac{(a_2-a_1)^{k+1}}{(k+1)!}\gamma^{(k)}(a_1),$$

$$I_{n,3}(\gamma) := \int_{a_1}^{a_2}\gamma(s)ds \cong \sum_{k=0}^{n-1}\frac{(-1)^k(a_2-a_1)^{k+1}}{(k+1)!}\gamma^{(k)}(a_2),$$

$$I_{n,4}(\gamma) := \int_{a_1}^{a_2}\gamma(s)ds \cong \sum_{k=0}^{n-1}\left[\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right](a_2-a_1)^{k+1}\gamma^{(k)}\left(\frac{a_1+a_2}{2}\right) + \frac{(-1)^{n+1}(a_2-a_1)^{n+1}}{2^n n!}\sigma_n,$$

$$I_{n,5}(\gamma) := \int_{a_1}^{a_2}\gamma(s)ds \cong \sum_{k=0}^{n-1}\left[\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right](a_2-a_1)^{k+1}\gamma^{(k)}\left(\frac{a_1+a_2}{2}\right) + \frac{(-1)^{n+1}(a_2-a_1)^{n+1}}{2^n n!}\sigma_n,$$

$$I_{n,6}(\gamma) := \int_{a_1}^{a_2}\gamma(s)ds \cong \sum_{k=0}^{n-1}\left[\frac{(a_2-a_1)^{k+1}}{(k+1)!}\right]\gamma^k(a_1) + \frac{(-1)^{n+1}(a_2-a_1)^{n+1}}{n!}\sigma_n,$$

$$I_{n,7}(\gamma) := \int_{a_1}^{a_2}\gamma(s)ds \cong \sum_{k=0}^{n-1}\left[\frac{(-1)^k(a_2-a_1)^{k+1}}{(k+1)!}\right]\gamma^k(a_2) + \frac{(-1)^{n+1}(a_2-a_1)^{n+1}}{n!}\sigma_n.$$

To demonstrate and compare the efficiency of the above mentioned quadrature rules we numerically integrate several functions with these quadrature rules and give their results in table 1 with the corresponding errors. The errors mentioned in table 1 is the absolute value of the difference of the exact value of the integral and its numerical value. All the quadrature rules report exact value of $\int_0^1 \gamma_1(s)ds$ for $n = 3$. This is because it is a polynomial of degree 2 and all it's higher derivatives are zero. For a polynomial of degree k, $n = k+1$ will give exact value of the integral. But acceptable error estimates can be obtained for smaller values of n. For $\int_0^1 \gamma(s)ds$, $I_{n,1}(\gamma)$ give an error of the order of 10^{-5} for $n = 5$ while the rest of the quadrature rules give a similar error for $n = 7$. Similarly for all other functions $I_{n,1}(\gamma)$ report errors of the order of 10^{-5} for relatively smaller values of n. Specifically, $I_{n,1}(\gamma)$ give an excellent estimate for $\int_0^1 \gamma(s)ds$ and $\int_0^1 \gamma_8(s)ds$ at $n = 2$ and $n = 3$ respectively.

In general $I_{n,1}(\gamma)$ gave better results as compared to the rest of the quadrature rules for much smaller values of n. Therefore we can conjecture that $I_{n,1}(\gamma)$ is computationally more efficient both in terms of error approximation, simplicity, and time. As a rough estimate we integrated

$$\gamma(s) = \log(s^2+2)\sin(\log(s+2))$$

using the built in algorithms of Mathematica 10.0 which took 26.30 seconds to give its approximate answer. To obtain similar approximation for $\int_0^1 \gamma(s)ds$, $I_{n,1}(\gamma)$ took less than a second. The performance of some the quadrature rules can be seen to be poor for $\int_0^1 \gamma_5(s)ds$ and $\int_0^1 \gamma_6(s)ds$ where we had to take n around 20 for $I_{n,2}(\gamma)$, $I_{n,3}(\gamma)$ and $I_{n,7}(\gamma)$ to achieve a reasonable approximation error. The reasons behind this need to be investigated.

Corollary 4 *If $\eta(s) \leq \gamma^{(n)}(s) \leq \xi(s)$ for any α and $\xi \in C[a_1,a_2]$ and $s \in [a_1,a_2]$, then by choosing $s = \frac{a_1+a_2}{2}$ in (8), the error of the midpoint type rule $I_{n,1}(\gamma)$ can have the following bounds*

$$\frac{1}{n!}\int_{a_1}^{\frac{a_1+a_2}{2}}(\theta-a_1)^n\eta(\theta)d\theta + \frac{1}{2n!}\int_{\frac{a_1+a_2}{2}}^{a_2}(\theta-a_2)^n \times [(\eta(\theta)+\xi(\theta)) + (-1)^{n+1}(\xi(\theta)-\eta(\theta))]d\theta$$
$$\leq (-1)^{n+1}\left(\sum_{k=0}^{n-1}\frac{[1+(-1)^k](a_2-a_1)^{k+1}}{2^{k+1}(k+1)!} \times \gamma^{(k)}\left(\frac{a_1+a_2}{2}\right) - \int_{a_1}^{a_2}\gamma(\theta)d\theta\right)$$

$$\leq \frac{1}{n!}\int_{a_1}^{\frac{a_1+a_2}{2}}(\theta-a_1)^n\xi(\theta)d\theta+\frac{1}{2n!}\int_{\frac{a_1+a_2}{2}}^{a_2}(\theta-a_2)^n \times[(\eta(\theta)+\xi(\theta))+(-1)^n(\xi(\theta)-\eta(\theta))]d\theta. \quad (19)$$

As a special case if we take $\eta(\theta)=\eta_0\neq 0$ and $\xi(\theta)=\xi_0\neq 0$, then the above inequality takes the following form

$$\frac{(a_2-a_1)^{n+1}}{2^{n+1}(n+1)!}\left(\alpha_0-\frac{1}{2}[\xi_0-\eta_0+(-1)^{n+1}(\eta_0+\xi_0)\right) \leq (-1)^{n+1}\left(\sum_{k=0}^{n-1}\frac{[1+(-1)^k](a_2-a_1)^{k+1}}{2^{k+1}(k+1)!}\times\gamma^{(k)}\left(\frac{a_1+a_2}{2}\right)-\int_{a_1}^{a_2}\gamma(\theta)d\theta\right) \leq \frac{(a_2-a_1)^{n+1}}{2^{n+1}(n+1)!}\left(\xi_0+\frac{1}{2}[\xi_0-\eta_0+(-1)^n(\eta_0+\xi_0)]\right) \quad (20)$$

provided that $\eta_0\leq\gamma^{(n)}(s)\leq\xi_0$ for all $s\in[a_1,a_2]$.

Corollary 5 *If the assumptions of Theorem 4 are satisfied and $s=a_1$ in (8), we get the following error bounds of nonstandard quadrature rule $I_{n,2}$*

$$\frac{1}{2n!}\int_{a_1}^{a_2}(\theta-a_2)^n[(\eta(\theta)+\xi(\theta))+(-1)^{n+1}(\xi(\theta)-\eta(\theta))]d\theta \leq (-1)^{n+1}\left(\sum_{k=0}^{n-1}\frac{(a_2-a_1)^{k+1}}{(k+1)!}\gamma^{(k)}(a_1)-\int_{a_1}^{a_2}\gamma(\theta)d\theta\right) \leq \frac{1}{2n!}\int_a^b(\theta-a_2)^n[(\eta(\theta)+\xi(\theta))+(-1)^n(\xi(\theta)-\eta(\theta))]d\theta. \quad (21)$$

provided that $\eta(s)\leq\gamma^{(n)}(s)\leq\xi(s)$ for any $s\in[a_1,a_2]$. Again as a special case if we take $\eta_0\leq\gamma^{(n)}(s)\leq\xi_0$, where η_0 and ξ_0 are non-zero constants then the following error bounds hold

$$\frac{(a_2-a_1)^{n+1}}{2(n+1)!}[(-1)^n(\eta_0+\xi_0)+\eta_0-\xi_0] \leq (-1)^{n+1}\left(\sum_{k=0}^{n-1}\frac{(a_2-a_1)^{k+1}}{(k+1)!}\gamma(a_1)-\int_{a_1}^{a_2}\gamma(\theta)d\theta\right) \leq \frac{(a_2-a_1)^{n+1}}{2(n+1)!}[\xi_0-\eta_0+(-1)^n(\eta_0+\xi_0)]. \quad (22)$$

Corollary 6 *If the assumptions of Theorem 4 are satisfied and $s=a_2$ in (8), we get the following error bounds of nonstandard quadrature rule*

$$\frac{1}{n!}\int_{a_1}^{a_2}(\theta-a_1)^n\eta(\theta)d\theta \leq (-1)^{n+1}\left(\sum_{k=0}^{n-1}\frac{(a_2-a_1)^{k+1}}{(k+1)!}\gamma(a_2)-\int_{a_1}^{a_2}\gamma(\theta)d\theta\right) \leq \frac{1}{n!}\int_{a_1}^{a_2}(\theta-a_1)^n\xi(\theta)d\theta. \quad (23)$$

Again as special case if we take $\eta_0\leq\gamma^{(n)}(\theta)\leq\xi_0$, η_0 and ξ_0 are non-zero constants then the following error bounds hold

$$\frac{(a_2-a_1)^{n+1}}{(n+1)!}\alpha_0\leq(-1)^{n+1}\times\left(\sum_{k=0}^{n-1}\frac{(-1)^k(a_2-a_1)^{k+1}}{(k+1)!}\gamma^{(k)}(a_2)-\int_{a_1}^{a_2}\gamma(\theta)d\theta\right)\leq\frac{(a_2-a_1)^{n+1}}{(n+1)!}\xi_0. \quad (24)$$

Corollary 7 *If $\eta(s)\leq\gamma^{(n)}(s)\leq\xi(s)$ for any $s\in[a_1,a_2]$ and $\eta,\xi\in C[a_1,a_2]$then the error bounds of the nonstandard quadrature rule $I_{n,4}(\gamma)$ can be bounded as*

$$\frac{1}{n!}\left(\int_{a_1}^{\frac{a_1+a_2}{2}}(\theta-a_1)^n\eta(\theta)d\theta+\int_{\frac{a_1+a_2}{2}}^{a_2}(\theta-a_2)^n\eta(\theta)d\theta\right)-\frac{(a_2-a_1)^n}{2^n n!}\int_{a_1}^{a_2}\eta(\theta)d\theta \leq (-1)^{n+1}\sum_{k=0}^{n-1}\left(\left[\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right](a_2-a_1)^{k+1}\times\gamma^{(k)}\left(\frac{a_1+a_2}{2}\right)-\int_{a_1}^{a_2}\gamma(\theta)d\theta\right)+\frac{(a_2-a_1)^n}{2^n n!}\left((a_2-a_1)\sigma_n-\int_{a_1}^{a_2}\eta(\theta)d\theta\right) \leq \frac{1}{n!}\int_{a_1}^{\frac{a_1+a_2}{2}}(\theta-a_1)^n\xi(\theta)d\theta+\int_{\frac{a_1+a_2}{2}}^{a_2}(\theta-a_2)^n\xi(\theta)d\theta-\frac{(a_2-a_1)^n}{2^n n!}\int_{a_1}^{a_2}\xi(\theta)d\theta. \quad (25)$$

Proof: In order to prove (25) we need to use the results of Theorem 5 and Theorem 6 simultaneously. By replacing $s=\frac{a_1+a_2}{2}$ in (13), we get

$$\frac{1}{n!}\left(\int_{a_1}^{\frac{a_1+a_2}{2}}(\theta-a_1)^n\eta(\theta)d\theta+\int_{\frac{a_1+a_2}{2}}^{a_2}(\theta-a_2)^n\eta(\theta)d\theta\right)+\frac{(a_2-a_1)^n}{2^n n!}\int_{a_1}^{a_2}\eta(\theta)d\theta \leq (-1)^{n+1}\left(\sum_{k=0}^{n-1}\left[\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right](a_2-a_1)^{k+1}\times\gamma^{(k)}\left(\frac{a_1+a_2}{2}\right)-\int_{a_1}^{a_2}\gamma(\theta)d\theta\right)+\frac{(a_2-a_1)^{n+1}}{2^n n!}\sigma_n, \quad (26)$$

where η for all $s \in [a_1, a_2]$.

Now by replacing $s = \frac{a_1+a_2}{2}$ in (16), we get

$$(-1)^{n+1}\left(\sum_{k=0}^{n-1}\left[\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right](a_2-a_1)^{k+1} \times \gamma^{(k)}\left(\frac{a_1+a_2}{2}\right) - \int_{a_1}^{a_2}\gamma(\theta)d\theta\right) + \frac{(a_2-a_1)^n}{2^n n!}\left(\gamma^{(n-1)}(a_2)-\gamma^{(n-1)}(a_1)\right)$$
$$\le \frac{1}{n!}\left(\int_{a_1}^{\frac{a_1+a_2}{2}}(\theta-a_1)^n\xi(\theta)d\theta + \int_{\frac{a_1+a_2}{2}}^{a_2}(\theta-a_2)^n\xi(\theta)d\theta\right) + \frac{(a_2-a_1)^n}{2^n n!}\int_{a_1}^{a_2}\xi(\theta)d\theta. \qquad (27)$$

Corollary 8 *If $\eta(s) \le \gamma^{(n)}(s) \le \xi(s)$ for any $s \in [a_1, a_1]$ and $\eta,\ \xi \in C[a_1, a_1]$, then the error bounds of the nonstandard quadrature $I_{n,5}$ rule are given as follows*

$$\frac{1}{n!}\left(\int_{a_1}^{\frac{a_1+a_2}{2}}(\theta-a_1)^n\xi(\theta)d\theta + \int_{\frac{a_1+a_2}{2}}^{a_2}(\theta-a_2)^n\xi(\theta)d\theta\right) - \frac{(a_2-a_1)^n}{2^n n!}\int_{a_1}^{a_2}\xi(\theta)d\theta$$
$$\le (-1)^{n+1}\left(\left[\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right] \times (a_2-a_1)^{k+1}\gamma^{(k)}\left(\frac{a_1+a_2}{2}\right) - \int_{a_1}^{a_2}\gamma(\theta)d\theta\right) + \frac{(a_2-a_1)^n}{2^n n!}\left((a_2-a_1)\sigma_n - \int_{a_1}^{a_2}\eta(\theta)d\theta\right)$$
$$\le \frac{1}{n!}\int_{a}^{\frac{a_1+a_2}{2}}(\theta-a_1)^n\eta(\theta)d\theta + \int_{\frac{a_1+a_2}{2}}^{a_2}(\theta-a_2)^n\eta(\theta)d\theta - \frac{(a_2-a_1)^n}{2^n n!}\int_a^b\eta(\theta)d\theta. \qquad (28)$$

Proof: The proof of (28) is similar to that of (25).

Corollary 9 *If $\eta(s) \le \gamma^{(n)}(s) \le \xi(s)$ for any $s \in [a_1, a_2]$ and $\eta,\ \xi \in C[a_1, a_2]$, then the error bounds of the nonstandard quadrature $I_{n,6}$ rule can have the following bounds*

$$\frac{1}{n!}\int_{a_1}^{a_2}[(\theta-a_2)^n-(a_2-a_1)^n]\xi(\theta)d\theta$$
$$\le (-1)^{n+1}\left(\sum_{k=0}^{n-1}\left[\frac{(a_2-a_1)^{k+1}}{(k+1)!}\right]\gamma^k(a_1) - \int_{a_1}^{a_2}\gamma(\theta)d\theta\right) + \frac{(a_2-a_1)^{n+1}}{n!}\sigma_n$$
$$\le \frac{1}{n!}\int_{a_1}^{a_2}[(s-a_2)^n-(a_2-a_1)^n]\eta(\theta)d\theta. \qquad (29)$$

Proof: The proof of (29) can be easily done by using (13) and (16) choosing $s = a_1$.

Corollary 10 *If $\eta(s) \le \gamma^{(n)}(s) \le \xi(s)$ for any $s \in [a_1, a_2]$ and $\eta,\ \xi \in C[a_1, a_2]$, then the error bounds of the nonstandard quadrature $I_{n,7}$ rule can be found as*

$$\frac{1}{n!}\int_{a_1}^{a_2}[(\theta-a_1)^n+(a_2-a_1)^n]\eta(\theta)d\theta$$
$$\le (-1)^{n+1}\left(\sum_{k=0}^{n-1}\left[\frac{(-1)^k(a_2-a_1)^{k+1}}{(k+1)!}\right] \times \gamma^k(s) - \int_{a_1}^{a_2}\gamma(\theta)d\theta\right) + \frac{(a_2-a_1)^{n+1}}{n!}\sigma_n$$
$$\le \frac{1}{n!}\int_{a_1}^{a_2}[(\theta-a_1)^n+(a_2-a_1)^n]\xi(\theta)d\theta. \qquad (30)$$

Proof: The error bounds given by (30) can be obtained from (13) and (16) for $s = a_2$.

Remark 1 *For $n = 1$ all the results established above become the results proved in [21]. The results from [21] give error bounds of the midpoint rule and some other nonstandard quadrature rules which depend upon the functions $\eta,\ \xi \in C[a_1, a_2]$ such that $\eta(s) \le \gamma'(s) \le \xi(s)$ for all $s \in [a_1, a_2]$ but our results can be used to find error bounds of many other new nonstandard quadrature rules for a particular choice of the natural number $n \ge 2$ which are expected to be very useful in numerical integration.*

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Table 1
Performance of the proposed quadrature rules

Method	$\boldsymbol{n}$, $I_{n,1}$	$\boldsymbol{n}$, $I_{n,2}$	$\boldsymbol{n}$, $I_{n,3}$	$\boldsymbol{n}$, $I_{n,4}$	$\boldsymbol{n}$, $I_{n,5}$	$\boldsymbol{n}$, $I_{n,6}$	$\boldsymbol{n}$, $I_{n,7}$	Exact
$\int_0^1 \gamma_1(s)ds$	**3**, 2.83333	**3**, 2.83333	**3**, 2.83333	**3**, 2.83333	**3**, 2.83333	**3**, 2.83333	**3**, 2.83333	2.83333
Error	0	0	0	0	0	0	0	
$\int_0^1 \gamma_2(s)ds$	**5**, 0.301153	**7**, 0.30119	**7**, 0.301021	**7**, 0.301163	**7**, 0.301174	**7**, 0.301905	**7**, 0.300307	0.301169
Error	1.5×10^{-5}	2.1×10^{-5}	1.4×10^{-4}	5.5×10^{-6}	5.65×10^{-6}	7.3×10^{-4}	8.4×10^{-4}	
$\int_0^1 \gamma_3(s)ds$	**5**, 0.909366	**7**, 0.909524	**7**, 0.909408	**7**, 0.909325	**7**, 0.909336	**7**, 0.910268	**7**, 0.908664	0.909331
Error	3.5×10^{-5}	1.9×10^{-5}	7.6×10^{-5}	5.5×10^{-6}	5.65×10^{-6}	9.3×10^{-4}	6.6×10^{-4}	
$\int_0^1 \gamma_4(s)ds$	**5**, 0.793033	**6**, 0.793056	**6**, 0.793182	**6**, 0.793042	**6**, 0.793023	**6**, 0.792417	**6**, 0.793821	0.793031
Error	1.48×10^{-6}	2.4×10^{-5}	1.5×10^{-4}	1.1×10^{-5}	8.49×10^{-6}	6.1×10^{-4}	7.8×10^{-4}	
$\int_0^1 \gamma_5(s)ds$	**7**, 1.46257	**11**, 1.46253	**15**, 1.46336	**10**, 1.46252	**10**, 1.46277	**19**, 1.4626	**19**, 1.46272	1.46265
Error	8.6×10^{-5}	1.2×10^{-4}	7.0×10^{-4}	1.2×10^{-4}	1.1×10^{-4}	5.3×10^{-5}	6.7×10^{-5}	
$\int_0^1 \gamma(s)ds$	**7**, 0.241593	**20**, 0.22908	**20**, 0.241572	**8**, 0.241592	**8**, 0.241593	**2**, 0.239583	**18**, 0.241601	0.241549
Error	4.3×10^{-5}	1.2×10^{-2}	2.3×10^{-5}	4.2×10^{-5}	4.3×10^{-5}	1.9×10^{-3}	5.1×10^{-5}	
$\int_0^1 \gamma_7(s)ds$	**2**, 1.3138	**11**, 1.31385	**23**, 1.31385	**11**, 1.3139	**11**, 1.31377	**18**, 1.31394	**18**, 1.31294	1.31383
Error	3.6×10^{-5}	1.4×10^{-5}	1.4×10^{-5}	6.5×10^{-5}	6.6×10^{-5}	1.0×10^{-4}	8.9×10^{-4}	
$\int_0^1 \gamma_8(s)ds$	**3**, 1.34102	**5**, 1.34167	**7**, 1.34149	**6**, 1.34149	**6**, 1.34146	**7**, 1.34138	**8**, 1.34145	1.34147
Error	4.5×10^{-4}	1.9×10^{-4}	1.9×10^{-5}	2.0×10^{-5}	1.5×10^{-5}	9.3×10^{-5}	2.2×10^{-5}	

$$\gamma_1(s) = s^2 + s + 2, \quad \gamma_2(s) = s\sin(s), \quad \gamma_3(s) = e^s\sin(s), \quad \gamma_4(s) = s^2 + \sin(s)$$

$$\gamma_5(s) = e^{s^2}, \quad \gamma_6(s) = \frac{1}{(s^4 + 4s^2 + 3)}, \quad \gamma_7(s) = e^s\cos(e^s - 2s), \quad \gamma_8(s) = \cos s + s.$$