CARISTI MAPPING IN MULTIPLICATIVE METRIC SPACES

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ABSTRACT: *The purpose of this paper is to define Caristi mapping in the setting of multiplicative metric space and prove fixed point theorems on multiplicative metric space endowed with a graph.*

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1 INTRODUCTION

Let (X, d) be a
metric space. A mapping $T : X \to X$ is said to be a Caristi mapping, [1] if there exists lower semi continuous function $\xi : X \to [0,\infty)$ satisfying $d(x,Tx) \leq \xi(x)-\xi(Tx)$ for each $x \in X$. Note that that each Caristi mapping on a complete metric space has a fixed point. Kirk [2] proved that the metric space (X, d) is complete if and only if each Caristi mapping on (X, d) has a fixed point.

Jachymaski [3] introduced the notion of Banach G-contraction and proved some fixed point theorems for mappings satisfying this notion on complete metric space with a graph. Several authors appreciated this novel work and proved several results on metric space with a graph see for example: [4-13].

Grossman and Katz [14] developed the new calculus called multiplicative (or non-Newtonian) calculus. Due to this calculus, Bashirov *et al.* [15] introduced the notion of multiplicative metric. That is, a mapping $m : X \times X \to [1,\infty)$ is called a multiplicative metric [15] on a nonempty set X if for each $x, y, z \in X$, m satisfies these conditions: (m_1): $m(x, y) > 1$ for all $x, y \in X$ and $m(x, y) = 1$ if and only if $x = y$; (m_2): $m(x, y) = m(y, x)$ for all $x, y \in X$; (m_3): $m(x, z) \leq m(x, y) \cdot m(y, z)$ for all $x,y,z \in X$.

Ozavsar and Cevikel [16] investigated the multiplicative metric spaces along with its topo-
logical properties, few of them are given below:

Let (X, m) is a multiplicative metric space. A sequence $\{x_n\}$ is said to be a multiplicative convergent to $x \in X$ denoted by $x_n \to^m x$, if for each $\varepsilon > 1$, there exists some $n_0 \in$ N such that $m(x_n, x) < \varepsilon$ for each $n \geq n_0$. A sequence $\{x_n\}$ is said to be a multiplicative Cauchy, if for each $\varepsilon > 1$, there exists $n_0 \in N$ such that $m(x_m, x_n) \leq \varepsilon$ for each $m, n \geq n_0$. A multiplicative metric space (X, m) is complete if every multiplicative Cauchy sequence in it is multiplicative convergent to some $x \in X$, [16].

Lemma 1.1. [16] Let (X, m) is a multiplicative metric space and $\{x_n\}$ is a sequence in X such that $x_n \to^m x$ and $x_n \to^m y$, then $x = y$.

In this paper, we extend the Caristi mapping in the setting of a multiplicative metric space. We prove fixed point theorems for such mappings on multiplicative metric space endowed with a graph G. We also construct an example to support our result.

2 MAIN RESULTS

Throughout this section, we assume that (X, m) is a multiplicative metric space and $G = (V, E)$ is a directed graph such that $V = X$, $\{(x, x): x \in V\} \subset E$ and G has no parallel edges.

Theorem 2.1. Let (X, m) be a complete multiplicative metric space endowed with the graph G. Let $T: X \to X$ be an edge preserving mapping such that for each $(x, Tx) \in E$, we have

$$m(x, Tx) \leq \frac{\xi(x)}{\xi(Tx)} \tag{2.1}$$

where $\xi : X \to [1,\infty)$ be any function. Further, assume that the following conditions hold:

(i) there exists $x_0 \in X$ such that $(x_0,Tx_0) \in E$;

(ii) T is G-continuous with respect to m, that is, $Tx_n \to^m Tx$ whenever, $x_n \to^m x$ and $(x_n , x_{n+1}) \in E$ for each $n \in N$.

Then T has a fixed point.

Proof. By hypothesis (i), we have $x_0 \in X$ such that $(x_0, x_1) \in E$, where $x_1 = Tx_0$. From (2.1), we have

$$m(x_0, x_1) \leq \frac{\xi(x_0)}{\xi(x_1)}.$$

Since T is edge preserving mapping, then $(x_1, x_2) \in E$. Again from (2.1), we have

$$m(x_1, x_2) \leq \frac{\xi(x_1)}{\xi(x_2)}.$$

Continuing in the same way we get a sequence $\{x_n\}$ in X such that $(x_n, x_{n+1}) \in E$ and

$$m(x_n, x_{n+1}) \leq \frac{\xi(x_n)}{\xi(x_{n+1})} \quad \text{for each } n \in \text{N}. \tag{2.2}$$

This implies that the sequence $\{\xi(x_n)\}$ is a nonincreasing sequence, which is bounded below by one, there exists $r \geq 1$ such that $\xi(x_n) \to^m r$. Now consider $m, n \in$ N, by using the multiplicative triangular inequality, we have

$$\begin{aligned} m(x_n, x_{m+n}) &\leq \prod_{i=n}^{n+m-1} m(x_i, x_{i+1}) \\ &\leq \prod_{i=n}^{n+m-1} \frac{\xi(x_i)}{\xi(x_{i+1})} \\ &= \frac{\xi(x_n)}{\xi(x_{n+m})}. \end{aligned} \tag{2.3}$$

This implies that $\{x_n\}$ is a Cauchy sequence in X, since $\xi(x_n) \to^m r$. By completeness of X, we have $x^* \in X$ such that $x_n \to^m x^*$. As T is G-continuous we have $Tx_n \to^m Tx^*$, that is, $x_{n+1} \to^m$

Tx^*. Since the multiplicative limit point is unique. Thus, $x^* = Tx^*$. □

Example 2.2. Let X = R be endowed with the multiplicative metric $m(x, y) = e^{|x-y|}$. The graph $G = (V, E)$ on X is defined as $V = X$ and $E = \{(x, y): x, y \geq 0\} \cup \{(x, x): x \in X\}$. Define the mapping $T: X \to X$ by

$$Tx = \begin{cases} x^2+1 & \text{if } x < 0 \\ x & \text{if } 0 \leq x < 1 \\ \sqrt{x} & \text{if } x > 1. \end{cases}$$

Define $\xi: X \to [1, \infty)$ by $\xi(t) = e^{2|t|}$ for each t. To see that, (2.1) holds, it is sufficient to consider the following cases:

(i) If $x \in [0,1)$, then for each $(x,Tx) \in X$, we have

$$m(x,Tx) = e^{|x-x|} = \frac{e^{2|x|}}{e^{2|x|}} = \frac{\xi(x)}{\xi(Tx)}.$$

(ii) If $x \geq 1$, then for each $(x,Tx) \in X$, we have

$$m(x,Tx) = e^{|x-\sqrt{x}|} < \frac{e^{2|x|}}{e^{2|\sqrt{x}|}} = \frac{\xi(x)}{\xi(Tx)}.$$

Thus, (2.1) holds. For $x_0 = 4$, we have $(x_0, Tx_0) \in E$. Moreover, T is G-continuous. Therefore, all conditions of Theorem 2.1 hold. Thus, T has fixed point.

In following theorem, we denote by $CL(X)$ the space of all multiplicative closed subsets of X. A mapping $T: X \to CL(X)$ is said to be an edge preserving if for each $u \in Tx$ and $v \in Ty$ we have $(u, v) \in E$, whenever $(x, y) \in E$.

Theorem 2.3. Let (X, m) be a complete multiplicative metric space endowed with the graph G.

Let $T: X \to CL(X)$ be an edge preserving mapping such that for each $x \in X$ and $y \in Tx$ with $(x, y) \in E$, there exists $z \in Ty$ satisfying

$$m(y,z) \leq \frac{\xi(x)}{\xi(y)} \tag{2.4}$$

where $\xi : X \to [1,\infty)$ be any function. Further, assume that the following conditions hold:

(i) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E$;

(ii) the mapping $g(x) = \inf\{m(x,a): a \in Tx\}$ is G-lower semi continuous, that is, for each sequence $\{x_n\}$ in X such that $x_n \to^m x$ and $(x_n, x_{n+1}) \in E$ for each $n \in N$, we have $g(x) \leq \liminf_{n\to\infty} g(x_n)$.

Then T has a fixed point.

Proof. By hypothesis (i), we have $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E$. From (2.4), we have $x_2 \in Tx_1$ such that

$$m(x_1,x_2) \leq \frac{\xi(x_0)}{\xi(x_1)}.$$

As T is edge preserving mapping, then $(x_1, x_2) \in E$. Again from (2.4), we have $x_3 \in Tx_2$ such that

$$m(x_2,x_3) \leq \frac{\xi(x_1)}{\xi(x_2)}.$$

Continuing in the same way we get a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$, $(x_n, x_{n+1}) \in E$ and

$$m(x_{n+1},x_{n+2}) \leq \frac{\xi(x_n)}{\xi(x_{n+1})}$$

for each $n \in$ N. (2.5)

This implies that sequence $\{\xi(x_n)\}$ is a nonincreasing sequence, which is also bounded below by one, there exists $r \geq 1$ such that $\xi(x_n) \to^m r$. Now consider $m, n \in$ N, by using the multiplicative triangular inequality, we have

$$\begin{aligned} m(x_n,x_{m+n}) &\leq \prod_{i=n}^{n+m-1} m(x_i,x_{i+1}) \\ &\leq \prod_{i=n}^{n+m-1} \frac{\xi(x_{i-1})}{\xi(x_i)} \\ &= \frac{\xi(x_{n-1})}{\xi(x_n)}. \end{aligned} \tag{2.6}$$

This implies that $\{x_n\}$ is a Cauchy sequence in X, since $\xi(x_n) \to^m r$. By completeness of X, we have $x^* \in X$ such that $x_n \to^m x^*$. Now, we have $\lim_{n\to\infty} m(x_n, x_{n+1}) = 1$. Thus by hypothesis (ii), we get $g(x^*) = \inf\{m(x^*, a): a \in Tx^*\} = 1$. Thus, $x^* \in Tx^*$. □

Corollary 2.4. Let (X, m) be a complete multiplicative metric space endowed with the graph G. Let $T: X \to X$ be an edge preserving mapping such that for each $(x, Tx) \in E$, we have

$$m(Tx,T^2x) \leq \frac{\xi(x)}{\xi(Tx)}$$

where $\xi : X \to [1,\infty)$ be any function. Further, assume that the following conditions hold:

(i) there exists $x_0 \in X$ such that $(x_0,Tx_0) \in E$;

(ii) T is G-continuous with respect to m, that is, $Tx_n \to^m Tx$ whenever, $x_n \to^m x$ and $(x_n, x_{n+1}) \in E$ for each $n \in N$.

Then T has a fixed point.

3 Consequence

In above theorems, if we assume that the graph $G = (V, E)$ is defined as $V = X$ and $E = \{(x,y) : x \preceq y\}$, then we get the following results:

Theorem 3.1. Let $(X,m,\preceq)$ be a complete ordered multiplicative metric space. Let $T: X \to X$ be an ordered preserving mapping such that for each $x \preceq Tx$, we have

$$m(x,Tx) \leq \frac{\xi(x)}{\xi(Tx)}$$

where $\xi : X \to [1,\infty)$ be any function. Further, assume that the following conditions hold:

(i) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;

(ii) T is ordered continuous with respect to m, that is, $Tx_n \to^m Tx$ whenever, $x_n \to^m x$ and $x_n \preceq x_{n+1}$ for each $n \in N$.

Then T has a fixed point.

Theorem 3.2. Let $(X,m,\preceq)$ be a complete ordered multiplicative metric space. Let $T: X \to CL(X)$ be an ordered preserving mapping such that for each $x \in X$ and $y \in Tx$ with $x \preceq y$, there exists $z \in Ty$ satisfying

$$m(y,z) \leq \frac{\xi(x)}{\xi(y)}$$

where $\xi : X \to [1,\infty)$ be any function. Further, assume that the following conditions hold:

(i) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $x_0 \preceq x_1$;

(ii) the mapping $g(x) = \inf\{m(x, a): a \in Tx\}$ is ordered-lower semi continuous function, that is, for each sequence $\{x_n\}$ in X such that $x_n \to^m x$ and $x_n \preceq x_{n+1}$ for each $n \in N$, we have $g(x) \leq \liminf_{n\to\infty} g(x_n)$.
Then T has a fixed point.

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