ON THE METRIC DIMENSION OF FAMILIES OF GRAPHS HAVING DIAMETER THREE

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ABSTRACT: *The concept of minimum resolving set has been proved to be useful and is related to a variety of fields such as chemistry [1,3], robotic navigation [2,5], combinatorial search, and optimization [4]. This work is devoted to evaluate the metric dimension of some families of graphs having diameter three.*

AMS Subject Classification: 05C12, 05C15, 05C78.
Key words: Diameter, Metric dimension, Resolving set.

1. INTRODUCTION

Resolving sets, in general graphs, were first studied by Harary, Melter [20], and Slater [22], although the resolving sets for hypercubes were studied earlier under the guise of a coin weighing problem [6,10,11,18,19]. Since then the resolving sets have been widely investigated, see for instance [9,12,13,14,15,17]. A resolving set arises also in many diverse areas including network discovery and verification [7], connected joins in graphs [21], and strategies for the mastermind games [8,16].

A *graph* G is an ordered pair (V, E), where V is the set of verteices and E the set of edges. The *distance* between vertices v,w ∈ V, denoted by d(v,w), is defined as the length of the shortest path between v and w, and the *diameter* of G, denoted by *dia*(G), is defined as the maximum distance among all pairs of vertices in G.

A vertex x ∈ V resolves a pair of vertices v,w ∈ V if $d(v, x) \neq d(w, x)$. A set of vertices W ⊆ V resolves G if each pair of distinct vertices of G is resolved by some vertex in W. The set W is called the *resolving set* of G if it resolves G. A resolving set W of G with the minimum cardinality is a *metric basis* for G, and the minimum cadinality is the metric dimension of G, which is denoted by β(G).

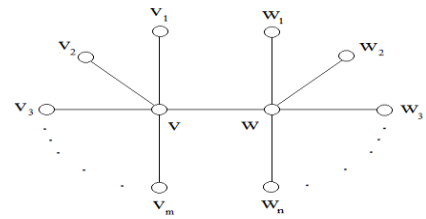
For an ordered subset W={ $w_1, w_2, ..., w_n$ } of vertices and a vertex *v* in a connected graph *G*, the representation of *v* with respect to *W* is the ordered *k*-tuple

$d(v|W) = (d(v,w_1), d(v,w_2), ..., d(v,w_n))$.

In this paper, we compute metric dimensions of some interesting families of graphs having diameter three.

2.MAIN RESULTS

The *bicentered graph*, denoted by $B_{m,n}$, is obtained from the path P_2 with vertices *v* and *w* by attaching *m* pendant vertices, $v_1, v_2, ..., v_m$, to the vertex *v* and *n* pendant vertices, $w_1, w_2, ..., w_n$, to the vertex *w* as you can see in the figure:



Theorem 1.

The metric dimension of $B_{m,n}$ is

$$\beta(B_{m,n}) = \begin{cases} 1 & ; \quad m = n = 1 \\ n & ; \quad m = 1, n > 1 \\ m + n - 2 & ; \quad m, n > 1. \end{cases}$$

Proof . The proof is divided into three cases:

Case I. (m=n=1) When m=n=1, the bicentered graph is simply the path P_4, which has metric dimension 1.

Case II. (m,n>1) Here we show that the resolving set is actually $W=\{v_1, v_2, ..., v_{m-1}, w_2, w_3, ..., w_n\}$. For this, take

$d(v_i | W) = (2, 2, ..., 2, 0, 2, ..., 2, 3, 3, 3, ..., 3)$,

for all $i = 1, 2, ..., m-1$, where 0 appears at *i*th place.

Also, $d(v_m | W) = (2, 2, 2, ..., 2, 3, 3, 3, ..., 3)$

$d(w_1 | W) = (3, 3, 3, ..., 3, 2, 2, 2, ..., 2)$, and

$d(w_i | W) = (3, 3, 3, ..., 3, 2, 2, 2, ..., 2, 2, 0, 2, 2, ..., 2)$,

for all $i = 2, 3, ..., n$, where 0 appears again at *i*th place.

Moreover, $d(v | W) = (1, 1, 1, ..., 1, 2, 2, 2, ..., 2)$ and

$d(w | W) = (2, 2, 2, ..., 2, 1, 1, 1, ..., 1)$.

Hence W is a resolving set.

To show that W has no proper resolving subset, firstly delete the vertex v_j from the set W where j=1,2,3,…,m-1 , then:

$d(v_j|W-\{v_j\}) = d(v_m|W-\{v_j\})$. Hence W-{ v_j } is not a resolving set.

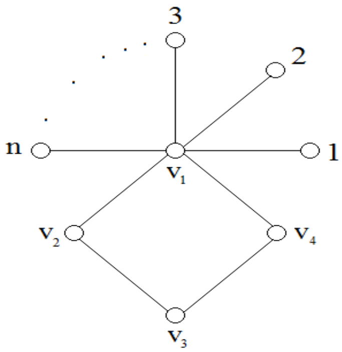
If we delete w_j from the set W where j=1,2,3,…,n then:

$d(w_j|W-\{w_j\}) = d(w_1|W-\{w_j\})$, and again $W-\{w_j\}$ is not a resolving set.

Thus, $\beta(B_{m,n}) = |W| = m+n-2$.

Case III. (m or n=1)
When any one of m or n is 1, the resolving set is $W=\{v_1, w_1, ..., w_{n-1}\}$. Due to similar reasons as give above, we get $\beta(B_{m,n}) = n.$ □

The second family of graphs we are interested in is $C_{4,n}$, which is obtained by attaching *n* pendant vertices to some vertex of C_4, as you can in the figure:



Theorem 2.
The metric dimension of the graph $C_{4,n}$ is n+1.

Proof. Here we show that the resloving set with minimum cardinality is W={1,2,3,…,n, v_2 }. Note that

$$d(i|W) = (2,2,2,...,2,2,0,2,2,...,2),$$

for all $i = 1,2,3,...,n.$ Also

$$d(v_1|W) = (1,1,1,...,1),$$

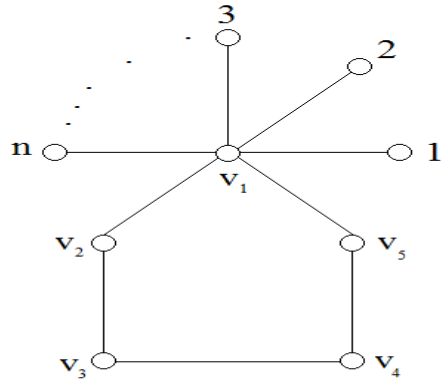
$$d(v_2|W) = (2,2,2,...,2,0),$$

$$d(v_3|W) = (3,3,3,...,3,1), \text{ and}$$

$$d(v_4|W) = (2,2,2,...,2).$$

Hence, W is a resolving set.
If we delete anyone vertex from W, then it does not remains a resolving set. For if we delete one of the n vertices from it, then $d(i|W) = d(v_4|W) = (2,2,2,...,2_n,2)$. $W-\{v_2\}$ is also not a resolving set because $d(v_2|W) = d(v_4|W) = (2,2,2,...,2_n)$. Since W has no proper resolving subset, $\beta(C_4(n)) = |W| = n+1$. □

The third family of graphs we discussed is $C_{5,n}$:



Theorem 3. The metric dimension of $C_{5,n}$ is:

$$\beta(C_{5,n}) = \begin{cases} 2; & n=1 \\ n; & n>1 \end{cases}$$

Proof. If n=1, then the resolving set is $W=\{1, v_3\}$. To prove W the resolving set, one can easily prove it by taking the distances of all the vertices with W. Take two proper subsets of W, $W_1 = \{1\}$ and $W_2 = \{v_3\}$. These two sets are not resolving sets because

$$d(v_3|W_1) = 3 = d(v_4|W_1)$$

and

$$d(v_1|W_2) = 2 = d(v_5|W_2)$$

respectively. So, W has no proper resolving subset. Hence, $\beta(C_{5,n}) = |W| = 2 \quad for \quad n=1$.

The resolving set of the $C_{5,n}; n>1$, with minimum cardinality is $W = \{1,2,3,...,n-1,v_3\}$. To prove W is a resolving set, $d(i|W) = (2,2,2,...,2,2,0,2,2,...,2,3),$ for all $i = 1,2,3,...,n-1,$ where 0 appears at *i*th place. Here

$$d(n|W) = (2,2,2,...,2,3),$$

$$d(v_1|W) = (1,1,1,....,1,2),$$

$$d(v_2|W) = (2,2,2,...,2,1),$$

$$d(v_3|W) = (3,3,3,...,3,0),$$

$$d(v_4|W) = (2,2,2,...,2), \text{ and}$$

$$d(v_5|W) = (3,3,3,...,3,1).$$

Hence W is a resolving set.
To prove W is minimum resolving set, we delete any vertex from W.

Case 1. If we first delete the i^{th} vertex from W, where i=1,2,…,n-1, then

$$d(i|W-\{i\}) = d(n|W-\{i\}) = (2,2,2,...,2,3).$$

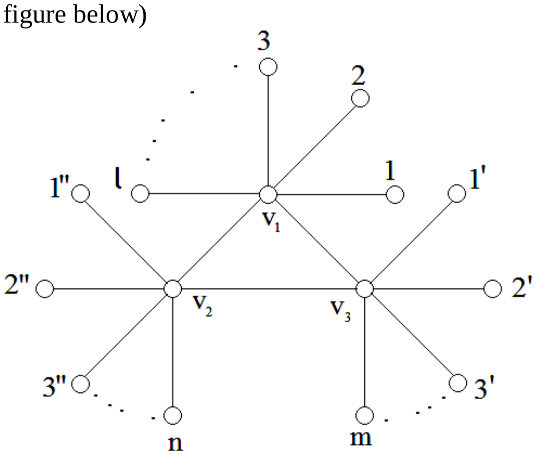
Hence, $W-\{i\}$ is not a resolving set.

Case 1I. Now, we delete v_3 vertex from the resolving set W.

$d(v_3|W-\{v_3\}) = d(v_5|W-\{v_3\}) = (3,3,3,...,3)$ Hence, $W-\{v_3\}$ is also not a resolving set,which implies

$\beta(C_5(n)) = |W| = n.$ □

The fourth family of graphs we work with is $C_{3,l,m,n}$. Let we have a cyclic graph C_3 with vertices v_1, v_2, v_3. The graph $C_{3,l,m,n}$ is obtained by attaching *l* pendant vertices $1,2,3,...,l$ with v_1, *m* pendant vertices $1',2',3',....,m$ at v_3, and *n* pendant vertices $1'',2'',3'',...,n$ at v_2.(as in figure below)



Theorem 4.

The dimension of a graph $C_{3,l,m,n}$, is

$$\beta(C_{3,l,m,n}) = \begin{cases} 2 \; ; & l=m=n=1 \\ (l-1)+(m-1)+(n-1) \; ; & otherwise \end{cases}$$

Proof.

We make three cases in this proof.

Case (1):

If $l=m=n=1$

The set $l=m=n=1$

W={1,2} is a resolving set. to prove W is a resolving set:

$$d(1|W) = (0,3)$$
$$d(2|W) = (3,0)$$
$$d(3|W) = (3,3)$$
$$d(v_1|W) = (1,2)$$
$$d(v_2|W) = (2,1)$$
$$d(v_3|W) = (2,2)$$

Hence, W is a resolving set. now! we have to prove that W has no proper resolving subset, the proper subsets of W are {1} and {2} which are not resolving sets, hence

$$\beta(C_{3,l,m,n}) = |W| = 2 \quad for \quad l=m=n=1$$

Case (2):

Anyone of the l,m,n is greater than 1, then

In this case the resolving set W is

$W = \{1,2,3,...,l-1,1',2',3',...,(m-1)',1'',2'',3'',...,(n-1)''\}$

To prove, W is a resolving set, we have

$d(i|W) = (2,2,2,...,2,0,2,...,2,3,3,3,...,3)$, for all $i=1,2,3,...,l-1$, where 0 appears at ith place.

$d(l|W) = (2,2,2,...,2,3,3,3,...,3)$

$d(i'|W) = (3,3,3,...,3,2,2,2,...,2,0,2,...,2,3,3,3,...,3)$, for all $i=1,2,3,...,m-1$, where 0 appears at ith place.

$d(m|W) = (3,3,3,...,3,2,2,2,...,2,3,3,3,...,3)$,

$d(i''|W) = (3,3,3,...,3,2,2,2,...,2,0,2,...,2)$, for all $i=1,2,3,...,n-1$, where o appears at ith place.

$d(v_1|W) = (1,1,1,....,1,2,2,2,...2)$,

$d(v_2|W) = (2,2,2,...,2,1,1,1,...,1)$,

$d(v_3|W) = (2,2,2,...,2,1,1,1,...,1,2,2,2,...,2)$.

Hence W is resolving set.

It remains to prove minimality of W. For this for discuss following cases.

Case 1. If we delete any one vertex from W from , $1,2,3,...,l-1$ say i we have:

$d(i|W-\{i\}) = d(l|W-\{i\})$.

Hence $W-\{i\}$ is not resolving set anymore.

Case 2.

If we delete one of vertex from $1',2',3',...,(m-1)'$ say i' from W, we have:

$d(i'|W-\{i'\}) = d(n|W-\{i'\})$,

hence $W-\{i'\}$ is not resolving set.

Case 3.

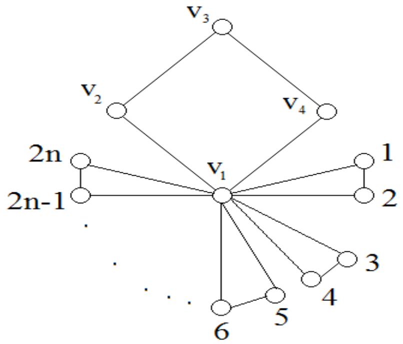
If we delete any one vertex from $1'',2'',3'',...,(n-1)''$ say i'' from W, then:

$d(i''|W-\{i''\}) = d(m|W-\{i''\})$

Hence, W has no proper subset which is a resolving set, thus $\beta(C_{3,l,m,n}) = |W| = (l-1)+(m-1)+(n-1);$ for $l,m,n>1$.

The fifth family of graphs we consider is $C_4 \circ C_3^n$.

This is obtained by joining n copies of C_3 at one vertex of C_4 as in figure below.



Theorem 5.

The metric dimension of a graph $C_4 \circ C_3^n$ is n+1.

Proof.

The set $W = \{1, 3, 5, 7, 2n-1, v_2\}$ is a resolving set, to prove W is a resolving set,

$$d(i \mid W) = (2, 2, 2, ..., 2, 1, 2..., 2, 2),$$

for $i = 1, 3, ..., 2n-1$, where 1 appears at *i*th place.

$$d(i \mid W) = (2, 2, 2, ..., 2, 1, 2, ..., 2, 2),$$

for $i = 2, 4, ..., 2n$, where 1 appears at *i*th place.

$$d(v_1 \mid W) = (1, 1, 1, ..., 1, 1),$$

$$d(v_2 \mid W) = (2, 2, 2, ..., 2, 0),$$

$$d(v_3 \mid W) = (3, 3, 3, ..., 3, 1),$$

$$d(v_4 \mid W) = (2, 2, 2, ..., 2, 2),$$

Hence, W is a resolving set.

Now, it remains to prove that W has no proper subset which is a resolving set. For this we consider following cases:

Case 1.

We delete any one vertex of W from 1,3,…,2n-1, say vertex I is deleted, then the set W-{i} is not a resolving set because

$$d(i \mid w\text{-}\{i\}) \;=\; d(v\_4 \mid w\text{-}\{i\}) \;=\; (2, 2, 2, ..., 2, 2)$$

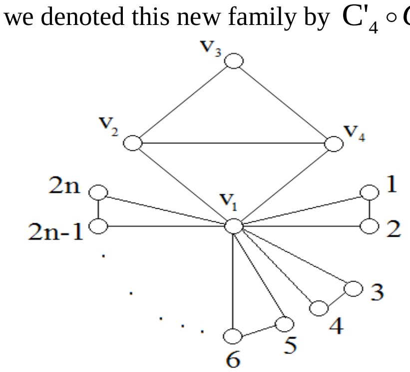
Case II.

Now, if we delete vertex v_2 from W, then the set W-{ v_2 } is again not a resolving set, because

$$d(v_2 \mid w\text{-}\{v_2\}) \;=\; d(v_4 \mid w\text{-}\{v_2\}) \;=\; (2, 2, 2, ..., 2.)$$

Hence W has no proper subset which is a resolving set. Since |W|=n+1 therefore, $\beta(C_4 \circ C_3{}^n) = n+1.$ □

The sixth family of graphs is obtained joining v_2 and v_4 of $C_4 \circ C_3^n$, we denoted this new family by $C'_4 \circ C_3^n$.



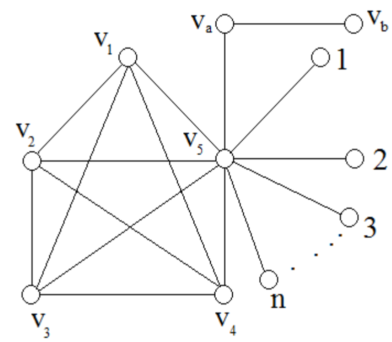
Theorem 6.

The metric dimension of the graph $C'_4 \circ C_3^n$ is also n+1.

Proof.

The prove is similar to the proof of $C_4 \circ C_3^n$.

The seventh family of graphs which we discussed is $K_5(2,n)$, which is obtained by attaching n pendent vertices 1,2,3,…,n and one brach having two vertices v_a and v_b at any one vertex of K_5 having vertices v_1, v_2, v_3, v_4, v_5, say at v_5. As in figure below



Theorem 7.

The metric dimension of a graph $K_5(2,n)$,

$$\beta(K_5(2,n)) \;=\begin{cases} 4; & n = 0 \\ n+3; & n > 0. \end{cases}$$

Proof.

Case 1. (When n=0) Let $W = \{v_2, v_3, v_4, v_5\}$, Since

$$d(v_1 \mid W) \;=\; (1,1,1,1)$$

$$d(v_2 \mid W) \;=\; (0,1,1,1)$$

$$d(v_3 \mid W) \;=\; (1,0,1,1)$$

$$d(v_4 \mid W) \;=\; (1,1,0,1)$$

$$d(v_5 \mid W) \;=\; (1,1,1,0)$$

$$d(v_a \mid W) \;=\; (2,2,2,1)$$

and

$$d(v_b \mid W) \;=\; (3,3,3,2)$$

Hence, W is the resolving set. To prove that W is minimal resolving set, let $W - \{v_i\}$ be any arbitrary subset of W, where v_i can be any one from v_2, v_3, v_4, v_5 then

$$d(v_i \mid W - \{v_i\}) \;=\; d(1 \mid W - \{v_i\}).$$

Hence, W is the resolving set with minimum cardinality, hence $\beta(K_5(2,n)) = 4 \quad if \quad n = 0.$

Case 2. (When n>0) In this case, take $W = \{v_2, v_3, v_4, 1, 2, 3, ..., n\}$, Since

$$d(i \mid W) = (2, 2, 2, 2, 2, ..., 2, 0, 2, ..., 2),$$

for i=1,2,…,n, where 0 appears at the *i*th place.

$d(v_1 | W) = (1,1,1,2,2,2,...,2),$

$d(v_2 | W) = (0,1,1,2,2,2,...,2),$

$d(v_3 | W) = (1,0,1,2,2,2,...,2),$

$d(v_4 | W) = (1,1,0,2,2,2,...,2),$

$d(v_a | W) = (2,2,2,...,2),$

and $d(v_b | W) = (3,3,3,...,3).$

Hence W is the resolving set for $K_5(a_n,b)$.

Now we have to prove that no subset of W is a resolving set. When we delete vertex i,(i=,1,2,3…,n), we receive W-{i} which is the subset of W, this is not a resolving set because:

$d(i | W - \{i\}) = d(a | W - \{i\}) = (2,2,2,...,2)$

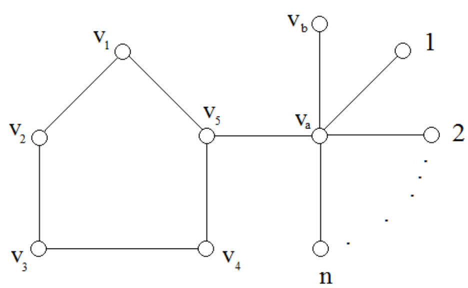
When we delete v_i where i=2,3,4, we receive $W - \{v_i\}$ which is not a resolving set, because:

$d(v_i | W - \{v_i\}) = d(v_1 | W - \{v_i\})$

Hence, W is a resolving set with minimal cardinality, thus:

$\beta(K_5(2,n)) = |W| = n+3$ for n>0.

The eight family of graphs is $K_5(a_n,b)$, which is shown in the figure below



Theorem 8.

The metric dimension of the graph $K_5(a_n,b)$, n+3.

Proof.

In this family of graphs, the resolving set is

$W = \{v_2, v_3, v_4, 1, 2, 3, ..., n\}$

because

$d(v_1 | w) = (1,1,1,3,3,3,...,3),$

$d(v_2 | w) = (0,1,1,3,3,3,...,3),$

$d(v_3 | w) = (1,0,1,3,3,3,...,3),$

$d(v_4 | w) = (1,1,0,3,3,3,...,3),$

$d(v_a | w) = (2,2,2,1,1,1,...,1),$

$d(v_b | w) = (3,3,3,2,2,2,...,2),$

and

$d(i | w) = (3,3,3,2,2,2,...,2,0,2,...,2),$

For i=1,2,3,…,n, where 0 appears at ith place.

Now we have to show that no other subset of W is a resolving set for this, we follow steps.

For the first step we delete the verrtex v_i from our resolving set W then

$d(v_i | W - \{v_i\}) = d(v_1 | W - \{v_i\})$

thus W-$\{v_i\}$ is not resolving set.

Now we remove the vertex *i* from W where *i* can be any one from the vertices 1,2,3,…,n, then

$d(v_a | W - \{j\}) = d(j | W - \{i\}),$

Thus W is the minimal resolving set

$\beta(K_5(a_n,b)) = |W| = n+3.$ □

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