HEISENBERG EQUATIONS FOR REAL SCALAR FIELD WITH $\lambda\phi^3$ INTERACTIONS

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ABSTRACT:*This paper presents a fractional Euler Lagrange equation for real scalar field with $\lambda\phi^3$ interaction. By applying the variational principle to a fractional action S, we obtained the fractional Euler–Lagrange equations of motion. Then we presented a Lagrangian and Hamiltonian densities for the fractional real scalar field with$\lambda\phi^3$ interaction of order α. We also provide expressions for the fractional Heisenberg equation for real scalar field with $\lambda\phi^3$ and compared it with Heisenberg equation for the classical field, in the limit α →1.*

Keywords*: Fractional Heisenberg equation, Left-Right PartialRiemann-Liouville and Caupto'sfractional derivative, fractional variational principle, fractional Lagrangian and Hamiltonian density*.

1-INTRODUCTION:

In modern physics, the physicists faced a problem cannot handle with it by traditional method, and it was necessary to find another procedure to solve it, and it was able using the fractional calculus [1-4]. Applications of fractional differential equations in physics have been spread rapidly, in particular condensed matter physics, where fractional differential equations are well suited to describe anomalous transport processes, such as anomalous diffusion, non-Debye relaxation process, etc. [5-12].

The Klein–Gordon equation of arbitrary fractional order and fractional Dirac equation have been studied by various authors [13-14]. Canonical quantization of fractional Klein–Gordon field has been considered by Amaral and Marino[13].We would like to point out that until now all these studies consider only free fractional fields. In this paper, we consider the real scalar fields with third degree self-interactionϕ^3.

Tarasov [15]considered a fractional derivative on a set of quantum observables as a fractional power of the commutator$\frac{i}{h}[H,O]$. As a result, he obtained a fractional generalization of the Heisenberg equation. The fractional Heisenberg equation is exactly solved for the Hamiltonians of free particle and harmonic oscillator. Rabei *et al.,* [16]applied fractional calculus to obtain the Hamiltonian formalism of non-conservative systems. They used the definition of Poisson bracket to obtain the equations of motion in terms of these brackets.The commutation relations and the Heisenberg equations of motion are also obtained.In this paper, we derived the fractional Euler-Lagrange equation for real scalar field with $\lambda\phi^3$ interaction by using the fractional variational principle. Furthermore,we aimto determine the fractional Heisenberg equation and compare it with the Heisenberg equation for classical fields.

The plan of this paper is as follows:

in Sect. 2, the definitions of Partial Left Right Riemann-Liouville and Caputo's fractional derivatives are discussed briefly. In Sect. 3, the fractional form of Euler- Lagrangian equation in terms of Left and Right Caputo's fractional derivative for real scalar field with $\lambda\phi^3$ interaction is presented. The fractional Heisenberg equation for real scalar field with$\lambda\phi^3$ interaction is investigated in Sect. 4. Sec. 5 closes the work with some concluding remarks.

2-Mathematical Tools

We have well-known definitions of a fractional derivative of order *α*>0 such as Riemann-Liouville, Grunwald-Letnikow, Caputo, and generalized functions approach. The most commonly used definitions are those of Riemann-Liouville and Caputo. We give some basic definitions of the fractional calculus theory, which are used throughout the paper.The left and the right partial Riemann-Liouville and Caputo fractional derivatives of order α_k, $0<\alpha_k<1$, of a function $f$depending on n variables, $x_1,\dots,x_n$ defined over the domain $\Omega=\prod_{i=1}^{n}[a_i,b_i]$ with respect to $x_k$are as follows [8]:

The Left Riemann-Liouvillefractional derivative

$$\left({}_{+}\partial_k^{\alpha}\right)(x) = \frac{1}{\Gamma(1-\alpha_k)}\partial x_k \int_{a_k}^{x_k} \frac{f\left(x_{1,}\dots,x_{k-1},u,x_{k+1,},\dots,x_{n,}\right)}{(x_k-u)^{\alpha_k}}du \qquad (1)$$

The Right Riemann-Liouvillefractional derivative

$$\left({}_{-}\partial_k^{\alpha}\right)(x) = \frac{-1}{\Gamma(1-\alpha_k)}\partial x_k \int_{x_k}^{b_k} \frac{f\left(x_{1,}\dots,x_{k-1},u,x_{k+1,},\dots,x_{n,}\right)}{(x_k-u)^{\alpha_k}}du \qquad (2)$$

The Left Caputo fractional derivative

$$({}_{+}^{C}\partial_k^{\alpha}f)(x) = \frac{1}{\Gamma(1-\alpha_k)}\int_{a_k}^{x_k}\frac{\partial_u f\left(x_{1,}\dots,x_{k-1},u,x_{k+1,},\dots,x_{n,}\right)}{(x_k-u)_k^{\alpha}}du \qquad (3)$$

The Right Caputo fractional derivative

$$({}_{-}^{C}\partial_k^{\alpha}f)(x) = \frac{-1}{\Gamma(1-\alpha_k)}\int_{a_k}^{x_k}\frac{\partial_u f\left(x_{1,}\dots,x_{k-1},u,x_{k+1,},\dots,x_{n,}\right)}{(u-x_k)_k^{\alpha}}du \qquad (4)$$

Where$\partial_{x_k}(g)$is the partial derivative of g with respect to the variablex_k. Here, in ${}_{+}\partial_k^{\alpha}$, ${}_{-}\partial_k^{\alpha}$, ${}_{+}^{c}\partial_k^{\alpha}$, ${}_{-}^{c}\partial_k^{\alpha}$, the subscript k and superscript α indicate that the derivative is taken with respect to the variable $x_k$and it is of the order α_k, the subscript + and – prior to the symbol ∂ represent the left and the right fractional derivatives, respectively.

The fractional variational principle and the Euler-Lagrange equation for field system described in terms of fractional derivatives presented in by use of a functional $S(\phi)$ asin [17].

$$S(\phi) = \int L[\phi(x_k),({}_{+}^{C}\partial_k^{\alpha})\phi(x_k),({}_{-}^{C}\partial_k^{\alpha})\phi(x_k),x_k](dx_k) \quad (5)$$

From this definition, we can obtain the fractional Euler-Lagrange equation as:

$$\frac{\partial L}{\partial \phi}+\sum_{k=1}^{n}{}_{-}\partial_k^{\alpha}\frac{\partial L}{\partial({}_{+}^{C}\partial_k^{\alpha}\phi)}+\sum_{k=1}^{n}{}_{+}\partial_k^{\beta}\frac{\partial L}{\partial({}_{-}^{C}\partial_k^{\beta}\phi)}=0 \quad (6)$$

Using the definition of the fractional canonical momentum density $\pi_{\alpha_k},\pi_{\beta_k}$ defined as $\pi_{\alpha_k}=\frac{\partial L}{\partial({}_{+}^{C}\partial_k^{\alpha}\phi)}$, $\pi_{\beta_k}=\frac{\partial L}{\partial({}_{-}^{C}\partial_k^{\beta}\phi)}$ then the fractional Euler Lagrange equation in terms of canonical momentum density takes the form

$$\frac{\partial L}{\partial \phi}+\sum_{k=1}^{n}{}_{-}\partial_k^{\alpha}\left(\pi_{\alpha_k}\right)+\sum_{k=1}^{n}{}_{+}\partial_k^{\beta}\left(\pi_{\beta_k}\right)=0 \quad (7)$$

Above equations (6, 7) are the Euler-Lagrange equations for the fractional field system and for $\alpha,\beta\rightarrow 1$ gives the usual Euler-Lagrange equation for classical fields.

3-Fractional Real Scalar Field with Interaction

Fractional lagrangian densities are functions of fractional fields and their derivatives. In this section we compute fractional Euler-Lagrange equation of the fractional real scalar field with $\lambda\phi^3$ interaction in a d-dimensional space-time. One of the classical Lagrangian where one does this is

$$\mathcal{L}=-\frac{\lambda}{3}\phi^3+\frac{1}{2}\left({}_{+}^{C}\partial_\mu^{\alpha}\phi\right)\left({}_{+}^{C}\partial_\mu^{\alpha}\phi\right)-\frac{m^{2\alpha}}{2}\phi^2 \quad (8)$$

Where $\lambda$is a coefficient that called coupling constant.

Using the fractional Euler Lagrange equation (6), it is easy to verify that this lagrangian density's corresponding equations of motion are

$$-\lambda\phi^2-m^{2\alpha}\phi+\sum_{\mu=1}^{n}{}_{-}\partial_\mu^{\alpha}\frac{\partial L}{\partial\left({}_{+}^{C}\partial_\mu^{\alpha}\phi\right)}=0 \quad (9)$$

Where

$$\begin{cases}\frac{\partial L}{\partial\phi}=-\lambda\phi^2-m^{2\alpha}\phi\\ \sum_{k=1}^{n}{}_{-}\partial_k^{\alpha}\frac{\partial L}{\partial({}_{+}^{C}\partial_k^{\alpha}\phi)}+\sum_{k=1}^{n}{}_{+}\partial_k^{\beta}\frac{\partial L}{\partial\left({}_{-}^{C}\partial_k^{\beta}\phi\right)}=\sum_{\mu=1}^{n}{}_{-}\partial_\mu^{\alpha}\frac{\partial L}{\partial({}_{+}^{C}\partial_\mu^{\alpha}\phi)}\end{cases}$$

If we restrict ourselves to space and time dimension

$$-\lambda\phi^2-m^{2\alpha}\phi+{}_{-}\partial_t^{\alpha}({}_{+}^{C}\partial_t^{\alpha}\phi)\ {}_{-}\partial_{x_i}^{\alpha}\left({}_{+}^{C}\partial_{x_i}^{\alpha}\phi\right)=0 \quad (10)$$

Forα→1, gives the usual Euler-Lagrange equations for real scalar field with$\lambda\phi^3$ interaction.Now we want to construct the Hamiltonian formulation for this real scalar field with the same degree of interaction.

The fractional Hamiltonian density defines as:

$$\mathcal{H}=\sum_{k=1}^{n}\pi_{\alpha_k}\left({}_{+}^{C}\partial_k^{\alpha}\phi\right)+\sum_{k=1}^{n}\pi_{\beta_k}\left({}_{-}^{C}\partial_k^{\alpha}\phi\right)-\mathcal{L} \quad (11)$$

This can be written as

$$\mathcal{H}=\frac{\partial\mathcal{L}}{\partial({}_{+}^{C}\partial_t^{\alpha}\phi)}\left({}_{+}^{C}\partial_t^{\alpha}\phi\right)-\mathcal{L} \quad (12)$$

Using the fractional canonical momentum density π

$$\pi=\frac{\partial\mathcal{L}}{\partial({}_{+}^{C}\partial_t^{\alpha}\phi)}={}_{+}^{C}\partial_t^{\alpha}\phi$$

Then, the Hamiltonian density becomes as

$$\mathcal{H}=({}_{+}^{C}\partial_t^{\alpha}\phi)({}_{+}^{C}\partial_t^{\alpha}\phi)-\left(\frac{1}{2}\left({}_{+}^{C}\partial_\mu^{\alpha}\phi\right)\left({}_{+}^{C}\partial_\mu^{\alpha}\phi\right)-\frac{m^{2\alpha}}{2}\phi^2-\frac{\lambda}{3}\phi^3\right) \quad (13)$$

This quantity can be written as

$$\mathcal{H}=({}_{+}^{C}\partial_t^{\alpha}\phi)({}_{+}^{C}\partial_t^{\alpha}\phi)-\left(\frac{1}{2}\left({}_{+}^{C}\partial_t^{\alpha}\phi\right)\left({}_{+}^{C}\partial_\mu^{\alpha}\phi\right)+\frac{1}{2}\left({}_{+}^{C}\partial_{x_i}^{\alpha}\phi\right)\left({}_{+}^{C}\partial_{x_i}^{\alpha}\phi\right)-\frac{m^{2\alpha}}{2}\phi^2-\frac{\lambda}{3}\phi^3\right) (14)$$

After subtract, we get

$$\mathcal{H}=\left(\frac{1}{2}\left({}_{+}^{C}\partial_t^{\alpha}\phi\right)\left({}_{+}^{C}\partial_\mu^{\alpha}\phi\right)+\frac{1}{2}\left({}_{+}^{C}\partial_{x_i}^{\alpha}\phi\right)\left({}_{+}^{C}\partial_{x_i}^{\alpha}\phi\right)+\frac{m^{2\alpha}}{2}\phi^2+\frac{\lambda}{3}\phi^3\right) \quad (15)$$

Substitute the canonical momentum density one can obtain the fractional Hamiltonian density as:

$$\mathcal{H}=\left(\frac{1}{2}\pi^2+\frac{1}{2}\left({}_{+}^{C}\partial_{x_i}^{\alpha}\phi\right)\left({}_{+}^{C}\partial_{x_i}^{\alpha}\phi\right)+\frac{m^{2\alpha}}{2}\phi^2+\frac{\lambda}{3}\phi^3\right) \quad (16)$$

Above equation is the Hamiltonian density for fractional real scalar field with $\lambda\phi^3$ interaction and for $\alpha\rightarrow 1$gives the usual Hamiltonian density for classical field.

4-Fractional Heisenberg Equation for Real Scalar Field with Interaction

To quantize the real scalar field, we interpret the field $\phi(t,x)$ and its conjugate momentum $\pi(t,x)$ as Hermitian operator in the Heisenberg Picture, and impose the usual equal timeCanonical commutation relations CCRs. The kronecker delta on a right side becomes a 3-dim Dirac delta function for the case of continuous field.

The fractional Heisenberg equation for the real scalar field with $\lambda\phi^3$interaction is

$${}_{+}^{C}\partial_t^{\alpha}\phi(t,x)=i\left[\hat{H},\hat{\phi}\right] \quad \text{, where} H=\int d^3x\hat{\mathcal{H}}$$

Then this quantity become as

$${}_{+}^{C}\partial_t^{\alpha}\phi(t,x)=i\left[\int d^3x\mathcal{H}\,,\hat{\phi}\right]$$

Substitute the quantity of Hamiltonian density

$${}_{+}^{C}\partial_t^{\alpha}\phi(t,x)=i\int d^3x\left[\left(\frac{1}{2}\pi^2+\frac{1}{2}\left({}_{+}^{C}\partial_{x_i}^{\alpha}\phi\right)\left({}_{+}^{C}\partial_{x_i}^{\alpha}\phi\right)+\frac{m^{2\alpha}}{2}\phi^2+\frac{\lambda}{3}\phi^3\right),\hat{\phi}\right] \quad (17)$$

Rearranging the last equation, we get

$${}_{+}^{C}\partial_t^{\alpha}\phi(t,x) = i\int d^3x\left\{\frac{1}{2}\left[\pi^2,\hat{\phi}\right] + \frac{1}{2}\left[\left({}_{+}^{C}\partial_{x_i}^{\alpha}\phi\right)^2,\hat{\phi}\right] + \frac{m^{2\alpha}}{2}\left[\phi^2,\hat{\phi}\right] + \frac{\lambda}{3}\left[\phi^3,\hat{\phi}\right]\right\} (18)$$

Since the last three terms vanish, then

$${}_{+}^{C}\partial_t^{\alpha}\phi(t,x) = i\int d^3x\left\{\frac{1}{2}\left[\pi^2(x,t),\hat{\phi}(x,t)\right]\right\} \quad (19)$$

Using the properties of canonicalcommutation relations, then the last equation take the forms respectively as in equations (20, 21, 22, 23)

$${}_{+}^{C}\partial_t^{\alpha}\phi(t,x) = \frac{i}{2}\int d^3x\left\{\pi(x,t)\left[\pi(x,t),\hat{\phi}(x,t)\right] + \left[\pi(x,t),\hat{\phi}(x,t)\right]\pi(x,t)\right\} (20)$$

$${}_{+}^{C}\partial_t^{\alpha}\phi(t,x) = \frac{i}{2}\int d^3x\left\{2\pi(x,t)\left[\pi(x,t),\hat{\phi}(x,t)\right]\right\} \quad (21)$$

$${}_{+}^{C}\partial_t^{\alpha}\phi(t,x) = i\int d^3x\{\pi(x,t)(-i)\delta^3(x-\acute{x})\} \quad (22)$$

$${}_{+}^{C}\partial_t^{\alpha}\phi(t,x) = -i^2\int d^3x\{\pi(x,t)\delta^3(x-\acute{x})\} \quad (23)$$

Using Delta-function properties, we get the first fractional Heisenberg equation

$${}_{+}^{C}\partial_t^{\alpha}\phi(t,x) = \pi(x,t) \quad (24)$$

Other equation of fractional Heisenberg equation, can be written as

$${}_{+}^{C}\partial_t^{\alpha}\pi(t,x) = i\left[\hat{H},\hat{\pi}\right] \quad (25)$$

Replace by $\mathcal{H}$ in equation (16),

$${}_{+}^{C}\partial_t^{\alpha}\pi(t,x) = i\int d^3x\left[\left(\frac{1}{2}\pi^2 + \frac{1}{2}\left({}_{+}^{C}\partial_{x_i}^{\alpha}\phi\right)\left({}_{+}^{C}\partial_{x_i}^{\alpha}\phi\right) + \frac{m^{2\alpha}}{2}\phi^2 + \frac{\lambda}{3}\phi^3\right),\hat{\pi}\right] \quad (26)$$

Rearranging the last equation

$${}_{+}^{C}\partial_t^{\alpha}\pi(t,x) = i\int d^3x\left\{\frac{1}{2}\left[\pi^2,\hat{\pi}\right] + \frac{1}{2}\left[\left({}_{+}^{C}\partial_{x_i}^{\alpha}\phi\right)^2,\hat{\pi}\right] + \frac{m^{2\alpha}}{2}\left[\phi^2,\hat{\pi}\right] + \frac{\lambda}{3}\left[\phi^3,\hat{\pi}\right]\right\} \quad (27)$$

The4-terms in last equation have the values as

$$\begin{cases} \frac{1}{2}\left[\pi^2,\hat{\pi}\right] = 0 \\ \frac{1}{2}\left[\left({}_{+}^{C}\partial_{x_i}^{\alpha}\phi\right)^2,\hat{\pi}\right] = {}_{+}^{C}\partial_{x_i}^{\alpha}\phi\, {}_{+}^{C}\partial_{x_i}^{\alpha} i\delta^3(x-\acute{x}) \\ \frac{m^{2\alpha}}{2}\left[\phi^2,\hat{\pi}\right] = m^{2\alpha} i\acute{\phi}\delta^3(x-\acute{x}) \\ \frac{\lambda}{3}\left[\phi^3,\hat{\pi}\right] = i\lambda\acute{\phi^2}\delta^3(x-\acute{x}) \end{cases}$$

Replace these values in equation (27), we obtain

$${}_{+}^{C}\partial_t^{\alpha}\pi(t,x) = i\int d^3x\left\{{}_{+}^{C}\partial_{x_i}^{\alpha}\phi\, {}_{+}^{C}\partial_{x_i}^{\alpha} i\delta^3(x-\acute{x}) + m^{2\alpha} i\acute{\phi}\delta^3(x-\acute{x}) + i\lambda\acute{\phi^2}\delta^3(x-\acute{x})\right\} (28)$$

Use the Delta –function properties

$${}_{+}^{C}\partial_t^{\alpha}\pi(t,x) = -\left({}_{+}^{C}\partial_{x_i}^{\alpha}\phi\right)^2 - m^{2\alpha}\acute{\phi} - \lambda\acute{\phi^2} \quad (29)$$

Since ${}_{+}^{C}\partial_t^{\alpha}\phi(t,x) = \pi(x,t)$

Then

$${}_{+}^{C}\partial_t^{2\alpha}\phi(t,x) = {}_{+}^{C}\partial_t^{\alpha}\pi(t,x) \quad (30)$$

Substitute the result in equation (29),

$${}_{+}^{C}\partial_t^{2\alpha}\phi(t,x) = -\left({}_{+}^{C}\partial_{x_i}^{\alpha}\phi\right)^2 - m^{2\alpha}\acute{\phi} - \lambda\acute{\phi^2} (31)$$

Rearrangement the last equation, we get

$${}_{+}^{C}\partial_t^{2\alpha}\phi(t,x) + \left({}_{+}^{C}\partial_{x_i}^{\alpha}\phi\right)^2 + m^{2\alpha}\acute{\phi} + \lambda\acute{\phi^2} = 0 \quad (32)$$

This can be written as

$$(D + m^{2\alpha})\acute{\phi} + \lambda\acute{\phi^2} = 0 \quad (33)$$

where D is the D'Almbertian operator.

Above equation is the equation for fractional real scalar field with $\lambda\phi^3$ interaction.

For α=1, the last equation reduced to the classical real scalar field equation with $\lambda\phi^3$ interaction.

5-Conclusion

There are different kinds of fractional variational calculus and fractional Euler-Lagrange equations due to the fact that we have several definitions for fractional derivatives. In this paper, we have proposed the fractional real scalar field with $\lambda\phi^3$ interaction. Then, using the fractional Euler-Lagrange equations, we have obtained fractional real scalar field equation with $\lambda\phi^3$ interaction. The generated results are found to be as the same as the real scalar field equation with $\lambda\phi^3$ interaction in the classical field when $\alpha \to 1$. In addition, we obtained the fractional Hamiltonian density for the real scalar field with $\lambda\phi^3$ interaction. In the second part of this paper it is observed that the Heisenberg equations of motion in fractional form can be constructed for real scalar field with $\lambda\phi^3$interaction. The generated results are found to be as the same as the real scalar field equation with $\lambda\phi^3$ interaction in the classical field when $\alpha \to 1$.

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