ON μ-SUPPLEMENTED AND COFINITELY μ-SUPPLEMENTED MODULES

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ABSTRACT. *Let R be a ring and M a right R- module. We extend the definition of supplemented module by replacing "small submodule" with "μ-small submodule" as we introduced in [1]. We show that any finite sum of μ-supplemented module is μ-supplemented , on the other hand we define and study the notion of amply , weakly , and cofinitely μ-supplemented modules . A module M is called ⊕-μ-supplemented module if every submodule of M has a μ-supplement which is a direct summand. The purpose of this work is to generalize supplemented modules and introduce various properties of these modules.*

Keywords. μ-supplemented, amply μ-supplemented , weakly μ-supplemented , cofinitely μ-supplemented and ⊕ - μ-supplemented .

1. INTRODUCTIOn.

Throughout this paper ,R is an associative ring with unity and all modules are unital right R- modules. A Submodule A of a module M is called small in M , written A<<M , if whenever M = A+B for any submodule B of M , we have M = B. See [2]. In [1] , we defined the notion of μ-small submodules as follows. A submodule A of M is called μ-small submodule , written $A<<_{\mu} M$ if whenever M = A+B with $\frac{M}{B}$ is cosingular , we have M = B. In this paper , we define a μ-supplemented module as a generalization of supplemented module as follows. A submodule A of M is called a μ-supplement of B in M if M = A+B and $A\cap B<<_{\mu} A$, if every submodule of M has μ-supplement , then M is called μ-supplemented module.

In section 2, we give some properties of μ-supplements , we prove that any factor module of μ-supplemented module is μ-supplemented and any finite sum of μ-supplemented modules is μ-supplemented.

In section 3, we introduced the notion of amply , weakly , and cofinitely μ-supplemented as a generalizations of amply ,weakly, and cofinitely supplemented modules. Recall that M is called amply supplemented if for any submodules A and B of M with M = A+B , A contains a supplement of B in M , see [3]. We call a module M is amply μ-supplemented if for any submodules A and B of M with M = A+B , A contains a μ-supplement of B in M. A module M is called weakly supplemented if for every submodule A of M , there is a submodule B of M such that M = A+B and $A\cap B<<M$, see [3]. We call a module M is weakly μ-supplemented if for every submodule A of M , there is a submodule B of M such that M = A+B and $A\cap B<<_{\mu} M$. Recall that a module M is called cofinitely supplemented if every cofinite submodule of M has a supplement submodule , [4]. We define the notion of cofinitely μ-supplemented as follows , a module M is called cofinitely μ-supplemented if every cofinite submodule of M has a μ-supplement.

In section 4, we introduced the concept of ⊕-μ-supplemented module as a generalization of ⊕-supplemented [5] as follows . The module M is called ⊕-μ-supplemented if every submodule of M has a μ-supplement which is a direct summand of M. Clearly ⊕-μ-supplemented modules are μ-supplemented and ⊕-supplemented are ⊕-μ-supplemented.

The aim of this work is to introduce μ- supplemented modules as a generalizations of supplemented modules , and some of it's generalizations , we state the main properties of μ- supplemented modules and introduced the main properties of μ- supplemented modules and supplying examples and remarks for these concepts. In this note, we answer the following natural question. Is any factor module of ⊕-μ-supplemented module is ⊕-μ-supplemented? In addition , we investigate direct summand of these modules.

2. μ-supplemented modules.

Definition 2.1. Let M be an R- module and let A , B be submodules of M, B is called μ- supplement of A in M , if M = A+B and $A\cap B <<_{\mu} B$. If every submodule of M has a μ- supplement , then M is called μ- supplemented module .

Recall that a module M is called μ-hollow if every proper submodule of M is μ-small in M , see [1].

Examples and Remarks2.1.

(1) Clearly that every μ-hollow module is μ-supplemented. For example Z_4 as Z-module. The converse is not true in general , for example , Z_6 as Z-module.

(2) Let A and B be submodules of an R- module M , if A is a μ-supplement of B , then B need not be a μ-supplement of A. For example , Z_4 as Z-module , Z_4 is a μ- supplement of $\{\bar{0},\bar{2}\}$ but $\{\bar{0},\bar{2}\}$ is not a μ-supplement of Z_4.

(3) M = A⊕B, then A is μ-supplement of B and B is μ-supplement of A, for example in Z_6 as Z- module $\{\bar{0},\bar{3}\}$ is μ-supplement of $\{\bar{0},\bar{2},\bar{4}\}$ and $\{\bar{0},\bar{2},\bar{4}\}$ is μ-supplement of $\{\bar{0},\bar{3}\}$.

(4) μ-supplement submodule need not be exists . For example , in Z as Z- module 2Z has no μ-supplement.

(5) It is clear that every supplemented module is a μ-supplemented. The converse is not true in general as the following example shows, Let $Q = \prod_{i=1}^{\infty} Fi$, where $F_i = Z_2$. Let R be the subring of Q generated by $\bigoplus_{i=1}^{\infty} F_i$ and 1_Q. Since R_R is μ-hollow module, hence it is μ-supplemented but not supplemented , see[1] and [6].

Proposition2.1. Let A be a μ-hollow submodule of the module M. Then A is a μ-supplement of each proper submodule B of M such that M = A+B.

Proof. Let B be a proper submodule of M such that M = A+B. Note $A \cap B \neq A$ if $A \cap B = A$, then $A \leq B$ implies that M = B which is a contradiction. Since A is μ-hollow , then $A \cap B <<_\mu A$. Thus A is a μ-supplement of B in M.

The following theorem gives a characterization of μ-supplement submodule.

Theorem2.1. Let A and B be submodules of an R-module M, then the following statements are equivalent.

(1) B is a μ-supplement of A in M.

(2) M = A+B and for every proper submodule X of B with $\frac{B}{X}$ is cosingular , then $M \neq A+X$.

Proof. (1)$\Rightarrow$(2) Assume that B is a μ-supplement of A in M and M = A+X , where X is a proper submodule of B such that $\frac{B}{X}$ is cosingular, then $B = B \cap M = B \cap (A+X) = X+(A \cap B)$, by modular law. Since B is a μ-supplement of A in M and $\frac{B}{X}$ is cosingular , then $A \cap B <<_\mu B$, hence B = X , which is a contradiction, because X is proper submodule of B. Thus $M \neq A+X$.

(2)$\Rightarrow$(1) Suppose that M = A+B , to prove that B is a μ-supplement of A in M , it is enough to show that $A \cap B <<_\mu B$, let U be a submodule of B such that $B = (A \cap B)+U$, $\frac{M}{U}$ is cosingular. If U is a proper submodule of B , then by our assumption $M \neq A+U$. But $M = A+B = A + (A \cap B) + U = A+U$, which is a contradiction. Thus B is a μ-supplement of A in M.

The following propositions gives some properties of μ-supplements.

Proposition2.2. Let A and B be submodules of an R-module M such that B is μ- supplement of A in M. Then

(1) If M = X+B , for some submodule X of A , then B is μ-supplement of X in M.

(2) If $C <<_\mu M$, then B is a μ-supplement of A+C.

(3) For any submodule Y of A , $\frac{(B+Y)}{Y}$ is a μ-supplement of $\frac{A}{Y}$ in $\frac{M}{Y}$.

Proof. (1) Assume that M = X+B , for some submodule X of A and B is a μ-supplement of A in M. Since $X \cap B \leq A \cap B <<_\mu B$, this implies that $X \cap B <<_\mu B$, by [1, Prop. 2.14]. But M = X+B , therefore, B is a μ-supplement of X.

(2) Let B be μ-supplement of A in M and $C <<_\mu M$. Clearly M = A+C+B. We show that $(A+C) \cap B <<_\mu B$, let $B = [(A+C) \cap B]+X$, for some submodule X of B , $\frac{B}{X}$ is cosingular. Then $M = (A+C) + B = (A+C)+[(A+C) \cap B]+X = C+ A+X$. Since $\frac{M}{A+X} = \frac{(A+X)+B}{(A+X)} \cong \frac{B}{B \cap (A+X)} = \frac{B}{(A \cap B)+X}$, by the second isomorphisim theorem , $\frac{B}{(A \cap B)+X}$ is cosingular by [1, Coro. 2.6], then $\frac{M}{A+X}$ is cosingular. But $C <<_\mu M$, therefore M = A+X. But B is a μ-supplement of A and $\frac{B}{X}$ is cosingular , therefore $B = B \cap (A+X) = X+(A \cap B) = X$. Thus, B is a μ-supplement of A+C.

(3) Let Y be a submodule of A , then $\frac{M}{Y} = \frac{(A+B)}{Y} = \frac{A}{Y} + \frac{(B+Y)}{Y}$. Also $\frac{A}{Y} \cap \frac{(B+Y)}{Y} = \frac{A \cap (B+Y)}{Y} = \frac{(A \cap B)+Y}{Y}$, by modular law. Now to show that $\frac{(A \cap B)+Y}{Y} <<_\mu \frac{(B+Y)}{Y}$. Let $\varphi: B \rightarrow \frac{(B+Y)}{Y}$ be a map defined by $\varphi(x) = x+Y$, for each $x \in B$. Clearly that φ is an epimorphisim. Since $A \cap B <<_\mu B$, then $\varphi(A \cap B) = \frac{(A \cap B)+Y}{Y} <<_\mu \frac{(B+Y)}{Y}$, by [1, Prop. 2.14]. Thus $\frac{(B+Y)}{Y}$ is a μ-supplement of $\frac{A}{Y}$ in $\frac{M}{Y}$.

Proposition2.3. Let M be an R-module and let A , B and C be submodules of M. Then.

(1) Assume that $M = M_1 \oplus M_2$. If A is a μ-supplement of A' in M_1 and B is a μ-supplement of B' in M_2, then $A \oplus B$ is a μ-supplement of $A' \oplus B'$ in M.

(2) If A is a μ-supplement of B in M and B is a μ-supplement of C in M , then B is a μ-supplement of A in M.

Proof. (1) By assumption, we have $M_1 = A + A'$ and $A \cap A' <<_\mu A$. Moreover, $M_2 = B + B'$ and $B \cap B' <<_\mu B$, then $M = (A \oplus B) + (A' \oplus B')$. By [1, Prop. 2.14], $(A \cap A') \oplus (B \cap B') <<_\mu A \oplus B$. One can easily show that $(A \oplus B) \cap (A' \oplus B') = (A \cap A') \oplus (B \cap B') <_\mu A \oplus B$, it follows that $A \oplus B$ is a μ-supplement of $A' \oplus B'$ in M.

(2) Let M = A+B = B+C , $A \cap B <<_\mu A$ and $B \cap C <<_\mu B$. We prove that $A \cap B <<_\mu B$. Let U be a submodule of B such that $B = (A \cap B)+U$, $\frac{B}{U}$ is cosingular , $M = B+C = (A \cap B)+U+C$. Since $A \cap B <<_\mu A$, then $A \cap B <<_\mu M$. Note that $\frac{M}{U+C} = \frac{B+C+U}{U+C} \cong \frac{B}{B \cap (U+C)} = \frac{B}{U+(B \cap C)}$ is cosingular. Since $A \cap B <<_\mu M$, then M = U+C. Now , $B = B \cap M = B \cap (U+C) = U+(B \cap C)$. But $B \cap C <<_\mu B$ and $\frac{B}{U}$ is cosingular , therefore B = U. Thus B is a μ-supplement of A in M.

To show that the sum of μ-supplemented modules is μ-supplemented , we need the following lemma.

Lemma2.1. Let M_1 and M_2 be submodules of M such that M_1 is μ-supplemented and $M_1 + M_2$ has a μ-supplement in M. Then M_2 has a μ-supplement in M.

Proof. By assumption, there exits a submodule A of M such that $M_1+M_2+A = M$ and $(M_1+M_2) \cap A <<_\mu A$. Moreover , since M_1 is μ-supplemented, $(M_2+A) \cap M_1$ has a μ-supplement in M_1 , there exists $B \leq M_1$ such that $M_1 = [(M_2+A) \cap M_1]+B$ and $(M_2 + A) \cap B <<_\mu B$. Then we have $M = M_1 + M_2 + A = [(M_2 + A) \cap M_1] + B + M_2 + A = M_2 + (B + A)$. One can easily show that $M_2 \cap (B + A) \leq [(M_2 + B) \cap A] + [(M_2 + A) \cap B] \leq [(M_2 + M_1) \cap A] + [(M_2 + A) \cap B] <<_\mu A+B$, by [1,Prop. 2.14] , it follows that $M_2 \cap (B + A) <<_\mu (B + A)$. Hence, B + A is a μ-supplement of M_2 in M. Thus M_2 has a μ-supplement in M.

Proposition2.4. Let M_1 and M_2 be μ-supplemented modules. If $M = M_1 + M_2$, then M is a μ-supplemented module.

Proof. Let A be a submodule of M. Since $M_1 + M_2 + A = M$ trivially has a μ-supplement 0 in M, $M_2 + A$ has a μ-supplement in M , by Lemma (2.6). Now, since M_2+A has a μ-supplemnet and M_2 is μ-supplemented , then A has a μ-supplement in M by Lemma (2.6) again. So M is a μ-supplemented module.

By induction , one can easily show that Any finite sum of μ-supplemented modules is μ-supplemented.

Proposition2.5. Epimorphic image of μ-supplemented module is μ-supplemented.

Proof. Let $f : M \rightarrow M'$ be an epimorphisim and let M be a μ-supplemented. To show that M' is a μ-supplemented , let A be a submodule of M' , $f^{-1}(A)$ is a submodule of M. But M is a μ-supplemented , therefore $f^{-1}(A)$ has a μ-supplement say B in M, hence $M = B+ f^{-1}(A)$ and $f^{-1}(A) \cap B \ll_\mu B$. Claim that $f(B)$ is μ-supplement of A. Since $M = B+ f^{-1}(A)$ and $f^{-1}(A) \cap B \ll_\mu B$, then $M' = f(B)+A$,and $A \cap f(B) \ll_\mu f(B)$, so $f(B)$ is μ-supplement of A. Thus M' is a μ-supplemented.

Corollary2.1. Every factor module of a μ-supplemented module is μ-supplemented .

Proof. Let M be a μ-supplemented module and let A be a submodule of M , let $\pi: M \rightarrow \frac{M}{A}$ be the natural epimorphisim. Since M is μ-supplemented , then $\frac{M}{A}$ is a μ-supplemented.

Remark. The converse of previous corollary is not true in general as the following example shows. Consider Z as Z- module , $6Z \leq Z$. Since $\frac{Z}{6Z} \cong Z_6$ is μ-supplemented but Z is not μ-supplemented module.

3. Amply , weakely and cofinitely μ-supplemented modules.

Definition3.1. Let M be an R- module. M is called amply μ-supplemented if for any submodules A and B of M with M = A+B , there exists a μ-supplement X of A contained in B.

Examples and Remarks3.1.

(1) Z_6 as Z- module is amply μ-supplemented.

(2) Z as Z- module is not amply μ-supplemented.

(3) Clearly that every amply supplemented module is amply μ-supplemented. The converse is not true in general , see [1,Example 3.17]. The converse hold when the module is cosingular.

(4) Every amply μ-supplemented is μ-supplemented.

Proposition3.1. Homomorphic image of amply μ-supplemented is amply μ-supplemented.

Proof. Let M be amply μ-supplemented and let $f : M \rightarrow M'$ be a homomorphisim , let A , B be submodules of M' such that M' = A+B , then $M = f^{-1}(A)+ f^{-1}(B)$. Since M is amply μ-supplemented , there exists μ-supplement X of $f^{-1}(A)$ in M which is contained in $f^{-1}(B)$, $M = f^{-1}(A)+X$ and $f^{-1}(A) \cap X \ll_\mu X$. Therefore , $M' = A+ f(X)$ and $A \cap f(X) = f(f^{-1}(A) \cap X) \ll_\mu f(X)$, by [1, Prop. 2.14]. Thus M' is amply μ-supplemented.

Corollary3.1. Let M be an amply μ-supplemented module and let A be a submodule of M , then $\frac{M}{A}$ is amply μ-supplemented.

Note. The converse of previous corollary is not true in general. For example , consider Z as Z- module $\frac{Z}{6Z} \cong Z_6$ is amply μ-supplemented but Z is not amply μ-supplemented.

A module M is said to be π-projective module , if for every two submodules A and B of M with M = A+B , there exists $f \in End(M)$ such that $Im f \leq A$ and $Im (I-f) \leq B$. See[3].

Proposition3.2. Let M be an R- module. If M is π-projective μ-supplemented , then M is amply μ-supplemented module.

Proof. Let M be a π-projective μ-supplemented and let A , B submodules of M such that M = A+B. Since M is π-projective , then there exists $f \in End(M)$ such that $Im f \leq A$ and $Im (I-f) \leq B$, A has a μ-supplement in M say C, then M = A+C and $A \cap C \ll_\mu C$. It can be seen that $M = f(M)+(I-f)(M) \leq A+ (I-f)(A+C) \leq A+(I-f)(C)$ and $A \cap (I-f)(C) \leq (I-f)(A \cap C) \ll_\mu (I-f)(C)$, therefore $(I-f)(C)$ is μ-supplement of A which is contained in B. Thus M is amply μ-supplemented.

Corollary3.2. Let M be an R- module. If M is projective μ-supplemented , then M is amply μ-supplemented module.

Proposition3.3. Let M be an R- module. If every submodule of M is μ-supplemented. Then M is an amply μ-supplemented.

Proof. Let A and B be submodules of M with M = A+B. By our assumption A is a μ-supplemented , $A \cap B$ has a μ-supplement say C in A, hence $(A \cap B)+C = A$ and $(A \cap B) \cap C = B \cap C \ll_\mu C$. Since $A = (A \cap B)+C \leq B+C$, hence M = B+C. Then C is a μ-supplement of B which is contained in A. Thus M is amply μ-supplemented.

Immediately , one can easily prove the following corollary.

Corollary3.3. Let R be any ring , then the following statements are equivalent.

(1) Every R- module is an amply μ-supplemented.

(2) Every R- module is μ-supplemented.

Definition3.2. Let M be an R- module. M is called weakly μ-supplemented module , if for each submodule A of M , there exists a submodule B of M such that M = A+B and $A \cap B \ll_\mu M$.

Examples and Remarks3.2.

(1) Z_4 as Z- module is weakly μ-supplemented.

(2) Clearly that every μ-supplemented module is weakly μ-supplemented , the converse is not true in general. For example Q as Z- module .

Let M be an R- module and let A be a submodule of M , recall that A is called a μ-coclosed submodule of M denoted by ($A \leq_{\mu cc} M$) if whenever $\frac{A}{X}$ is cosingular and $\frac{A}{X} \ll_\mu \frac{M}{X}$ for some submodule X of A , we have X = A. See [1].

Proposition3.4. Let A be a submodule of an R-module M. Consider the following statements.

(1) A is μ-supplement submodule of M .

(2) A is μ-coclosed in M.

(3) For every submodule X of A , if $X \ll_\mu M$, then $X \ll_\mu A$. Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3)

If M is weakly μ-supplemented , then (3) $\Rightarrow$ (1)

Proof. (1) $\Rightarrow$ (2) Let A be a μ-supplement of B in M , then M = A+B and $A \cap B \ll_\mu A$. To prove that A is μ-coclosed , assume that $\frac{A}{X}$ is cosingular and $\frac{A}{X} \ll_\mu \frac{M}{X}$, for some submodule X of A. Since M = A+B , $\frac{M}{X} = \frac{A}{X} + \frac{B+X}{X}$. We have $\frac{M}{B+X} = \frac{A+B+X}{B+X} \cong \frac{A}{A \cap (B+X)} = \frac{A}{X+(A \cap B)}$, which is cosingular by corollary [1]. But $\frac{A}{X} \ll_\mu \frac{M}{X}$, therefore $\frac{M}{X} = \frac{B+X}{X}$ which implies that M = B+X. Note that $A = A \cap M = A \cap (B+X) = X+(A \cap B)$, but $A \cap B \ll_\mu A$ and $\frac{A}{X}$ is cosingular , therefore A = X. Thus A is a μ-coclosed.

(2) $\Rightarrow$ (3) See [1, Prop. 3.3]

(3)$\Rightarrow$(1) Since M is weakly μ-supplemented , there exists a submodule B of M such that M =A+B and $A\cap B<<_{\mu}$ M. By (3) $A\cap B<<_{\mu}$ A. Thus A is μ-supplement submodule of M.

Definition3.3. An R- module M is called cofinitely μ-supplemented (briefly cof-μ-supplemented) if each cofinite submodule of M has a μ-supplement in M.

Examples and Remarks3.3.

(1) Z_6 as Z- module is cof-μ-supplemented.

(2) Z as Z- module is not cof-μ-supplemented , since 2Z is a cofinite submodule of Z which has no μ-supplement.

(3) It is clear that every μ-supplemented is cof-μ-supplemented. But the converse is not true in general as the following example shows , Q as Z- module is cof-μ-supplemented , since the only cofinite submodule of Q is Q which has a μ-supplement , but we know that Q is not μ-supplemented Z- module.

The following proposition gives a condition under which the μ-supplemented and cof-μ-supplemented are equivalent

Proposition3.5. Let M be a finitely generated R-module. Then M is μ-supplemented if and only if M is cof-μ-supplemented.

Proof. To show that M is μ-supplemented , let A be a submodule of M. Since M is finitely generated , then $\frac{M}{A}$ is finitely generated , hence A is cofinite submodule of M. But M is cof-μ-supplemented , therefore A has μ-supplement in M. Thus M is μ-supplemented. The converse is clear.

Next , we give some properties of cof-μ-supplemented modules.

To show that arbitrary sum of cof-μ-supplemented is cof-μ-supplemented , we need the following standard lemma.

Lemma3.1. Let A,B be submodules of a module M such that A is cof-μ-supplemented, B is cofinite in M and A+B has a μ-supplement C in M. Then A $\cap$ (B + C) has a μ-supplement X in A. Moreover, C + X is μ-supplement of B in M.

Proof. Let C a μ-supplement of A + B in M. Thus M = C + A + B and C $\cap$ (A + B)$<<_{\mu}$ C. Now $\frac{A}{A\cap(B+C)}\cong\frac{A+B+C}{B+C}=\frac{M}{B+C}\cong\frac{\frac{M}{B}}{\frac{B+C}{B}}$, which is finitely generated, hence $A\cap$(C + B) is cofinite in A. But A is cof-μ-supplemented, there exists a submodule X of A such that X is a μ-supplement of $A\cap$(C + B) in A. Thus A = X +[$A\cap$(C+B)] and $X\cap A\cap$(C +B) = $X\cap$(C+B)$<<_{\mu}$ X , so X is a μ-supplement of (B+C) in A. Now , to show that C +X is a μ-supplement of B in M, we have M = C +A+B = C +X +[$A\cap$(C +B)]+B = C +X + B, and one can easily show that $B\cap$(C +X) ≤ [$C\cap$(B +X)]+[$X\cap$(C+B)]$<<_{\mu}$ C +X. Therefore ,C + X is a μ-supplement of B in M.

Proposition3.6. An arbitrary sum of cof-μ-supplemented modules is cof-μ-supplemented.

Proof. Suppose that $\{M_i\}_{i\in I}$ is a family of cof-μ-supplemented modules, and let M = $\sum_{i\in I} Mi$. Let A be a cofinite submodule of M, so M = A+M_{i1} +· · ·+M_{in} for some $n \in$ N, $ik \in$ I. Since A is cofinite in M and M has a zero μ-supplement , then by previous lemma, (M_{i1} +..… +M_{in})$\cap$(0+A) has a μ-supplement say X in A, Moreover, X is a μ-supplement of A in M. Thus M is cof-μ-supplemented module.

Proposition3.7. Homomorphic image of a cof-μ-supplemented is cof-μ-supplemented.

Proof. Let f :M$\rightarrow$M' be a homomorphisim and M be a cof-μ-supplemented. To show that f (M) is cof-μ-supplemented , let A be a cofinite submodule of f (M) , hence f^{-1}(A) is cofinite submodule of M. But M is cof-μ-supplemented , therefore f^{-1}(A) has a μ-supplement say B in M , then M = f^{-1}(A) +B and f^{-1}(A)$\cap$B$<<_{\mu}$ B. Hence f(M) = A+ f(B) and A$\cap f$(B) $<<_{\mu}$ f(B) , that is A is μ-supplement of f (B). Thus f (M) is a cof-μ-supplemented.

Corollary3.4. Let M be a cof-μ-supplemented , let A be a submodule of M , then $\frac{M}{A}$ is cof-μ-supplemented.

Proof. Let M be a cof-μ-supplemented and let π: M$\rightarrow\frac{M}{A}$ be the natural epimorphisim , by previous proposition $\frac{M}{A}$ is cof-μ-supplemented.

The converse is not true in general , for example , consider Z as Z- module $\frac{Z}{6Z}\cong Z_6$ is cof-μ-supplemented but Z is not cof-μ-supplemented.

Corollary3.5. Let M be a cof-μ-supplemented , then any direct summand of M is cof-μ-suplemented.

Proof. Let M be a cof-μ-supplemented , D be a direct summand of M and let P: M$\rightarrow$D be the projection epimorphisim. By proposition (3.17) D is cof-μ-supplemented.

4. $\oplus$ -μ-supplemented modules

Definition4.1. A module M is called $\oplus$-μ-supplemented module if every submodule of M has a μ−supplement which is a direct summand of M.

Examples and Remarks4.1.

(1) Every semisimple is $\oplus$-μ-supplemented. For example , Z_6 as Z- module is $\oplus$-μ-supplemented.

(2) Z as Z- module is not $\oplus$-μ-supplemented.

(3) Clearly that every $\oplus$-μ-supplemented is μ-supplemented and every $\oplus$-supplemented is $\oplus$-μ-supplemented.

But the converse is not true in general. See example (2.2-5)

Now , consider Z_4 as Z- module clearly that Z_4 is μ-supplemented but not $\oplus$-μ-supplemented.

An R− module M is said to have property (D3), if M_1 and M_2 are direct summands of M with M = M_1+M_2, then $M_1\cap M_2$ is also a direct summand of M [5].

Proposition4.1. Let M be a $\oplus$-μ-supplemented module with (D3). Then every direct summand of M is a $\oplus$-μ-supplemented module.

Proof. Let M be a $\oplus$-μ-supplemented with (D3) and let A be a direct summand of M. To show that A is a $\oplus$-μ-supplemented, let X be a submodule of A. Then there exists a direct summand Y of M such that Y is μ-supplement of X, then M = X+Y and $X\cap Y <<_{\mu}$ Y. Since X ≤ A , M = A+Y . Since A and Y are direct summands of M and M = A+Y , $A\cap Y$ is a direct summand of M and hence it is a direct summand of A , because M satisfy D3. By modularity, we have A = A

$\cap$ M = A $\cap$ (X+Y) = X+(A $\cap$ Y). Note that X $\cap$ (A $\cap$ Y) = X $\cap$ Y $<<_{\mu}$ Y . But , A $\cap$ Y is a direct summand of M , therefore X $\cap$ Y $<<_{\mu}$ A $\cap$ Y , by [1, Prop. 2.15]. Thus A is a $\oplus$ −μ-supplemented module.

Let M be an R- module and let A be a submodule of M, A is called fully invariant submodule of M if f (A) ≤ A , for each $f \in$ End(M). Let M be an R- module . Recall that M is called a duo module if every submodule of M is fully invariant. See [7].

Proposition4.2. Let M be a $\oplus$-μ-supplemented module and A be a fully invariant submodule of M. If A is a direct summand of M, then A is a $\oplus$-μ−supplemented module.

Proof. Let A be a direct summand of M and X be a submodule of A. Since M is a $\oplus$-μ-supplemented , there exist a direct summand Y of M, such that M = X + Y , X $\cap$ Y $<<_{\mu}$ Y and M = Y $\oplus$ Y' , Y'≤M. we have A = A $\cap$ M = A $\cap$ (Y $\oplus$ Y') = (A $\cap$ Y) $\oplus$ (A $\cap$ Y') , because A is a fully invariant submodule of M. If we show that A $\cap$ Y is μ-supplement of X in A, then the proof is complete. Since M = X + Y, we have A = A $\cap$ M = A $\cap$ (X +Y) = X+ (A $\cap$ Y). Now , X $\cap$ Y $<<_{\mu}$ M. Due to A $\cap$ Y is a direct summand of M, we obtain X $\cap$ Y $<<_{\mu}$ A $\cap$ Y by [1, Prop.2.15]. Hence A $\cap$ Y is a μ−supplement of X in A which is a direct summand of A. So it implies that A is a $\oplus$-μ-supplemented module.

The following theorem shows that the direct sum of $\oplus$-μ-supplemented modules is $\oplus$-μ-supplemented.

Theorem4.1. Let M_1 and M_2 be a $\oplus$-μ-supplemented modules. If M = $M_1 \oplus M_2$, then M is a $\oplus$-μ-supplemented module.

Proof. Let A be any submodule of M. Then M = M_1 + M_2 + A and so M_1+M_2+A has a $\oplus$-μ-supplement 0 in M. Since M_1 is a $\oplus$-μ-supplemented module, $M_1 \cap (M_2+A)$ has a μ-supplement X in M_1, then we have $M_1 = [M_1 \cap (M_2+A)]+X$ and $M_1 \cap (M_2+A) \cap X = (M_2+A) \cap X <<_{\mu} X$ such that X is direct summand of M_1. Claim that X is a μ-supplement of M_2+A in M. Since $M_1 = [M_1 \cap (M_2+A)]+X$, then $M = M_1+M_2 = [M_1 \cap (M_2+A)]+X+A+M_2 = X+A+M_2$ and $X \cap (M_2+A) = M_1 \cap (M_2+A) \cap X <<_{\mu} X$, hence X is a μ-supplement of M_2+A in M. Now , since $M_2 \cap (A+X) \leq M_2$ and M_2 is $\oplus$-μ-supplemented , then $M_2 \cap (A+X)$ has a μ-supplement Y in M_2 and Y is a direct summand of M_2 , then we have , $M_2 = Y \oplus Y'$, $Y' \leq M_2$, $M_2 = M_2 \cap (A+X)+Y$ and $M_2 \cap (A+X) \cap Y = (A+X) \cap Y <<_{\mu} Y$. Since $M = M_2+A+X = Y+M_2 \cap (A+X)+(A+X) = Y+A+X$ and $X \cap (Y+A) \leq X \cap [Y+[M_2 \cap (A+X)]+A] \leq X \cap (M_2+A) <<_{\mu} X$ and $M_2 \cap (A+X) \cap Y = Y \cap (A+X) <<_{\mu} Y$. One can easily show that $A \cap (X+Y) \leq X \cap (Y+A) + Y \cap (A+X) <<_{\mu} X+Y$. So , X+Y is μ-supplement of A in M. Thus M is $\oplus$-μ-supplemented.

Corollary4.1. Any finite direct sum of $\oplus$-μ-supplemented modules is $\oplus$-μ-supplemented module.

Proof. By induction.

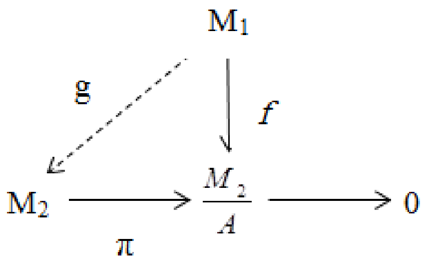
Let M_1 and M_2 be R- modules. Recall that M_1 is M_2-projective if for every submodule A of M_2 and any homomorphisim f : $M_1 \rightarrow \frac{M_2}{A}$, there is a homomorphisim g : $M_1 \rightarrow M_2$ such that $\pi \circ g = f$, where π : $M_2 \rightarrow \frac{M_2}{A}$ is the natural epimorphisim , see [8].

M_1 and M_2 are said to be relatively projective if M_1 is M_2- projective and M_2 is M_1- projective.

Theorem4.2. Let M_i ($1 \leq i \leq n$) be any finite collection of relatively projective modules. The module M = $M_1 \oplus M_2 \oplus \ldots \oplus M_n$ is a $\oplus$-μ-supplemented module if and only if M_i is a $\oplus$-μ-supplemented module for each $1 \leq i \leq n$.

Proof. The necessity part is proved in Theorem 4.5 .

Conversely , it is sufficient to prove that M_1 is $\oplus$-μ-supplemented. Let A be any submodule of M_1. Then there exist a direct summand B of M such that M = A+B = B $\oplus$ B' and A $\cap$ B $<<_{\mu}$ B. Note that M = A +B



= M_1 + B. By [8,Lemma 4.47], there exists a submodule B_1 of B such that M = $M_1 \oplus B_1$. Now , B = B $\cap$ M = B $\cap (M_1 \oplus B_1) = B_1 \oplus (B \cap M_1)$, then $(B \cap M_1)$ is a direct summand of M and hence it is a direct summand of M_1. Now , we have $M_1 = M_1 \cap M = M_1 \cap (A+B) = A+(B \cap M_1)$ and $A \cap B \cap M_1 = A \cap B <<_{\mu} B$, $A \cap B \cap M_1 <<_{\mu} B \cap M_1$, because $B \cap M_1$ is a direct summand of M. Therefore , $B \cap M_1$ is μ-supplement of A in M_1 which is a direct summand. Thus A is $\oplus$-μ-supplemented.

Proposition4.3. Let M be a nonzero $\oplus$-μ-supplemented module and let A be a fully invariant submodule of M. Then the factor module $\frac{M}{A}$ is a $\oplus$-μ-supplemented.

Proof. To show that $\frac{M}{A}$ is $\oplus$-μ-supplemented , let $\frac{B}{A}$ be any submodule of $\frac{M}{A}$. Since M is $\oplus$-μ-supplemented module, there exist a direct summand C and of M such that M = C + B, B $\cap$ C $<<_{\mu}$ C and M = C $\oplus$ C' , C' ≤ M. By proposition (2.5) $\frac{C+A}{A}$ is μ-supplement of $\frac{B}{A}$ in $\frac{M}{A}$. Since A is a fully invariant submodule of M , then $\frac{C+A}{A}$ is a direct summand of $\frac{M}{A}$. Thus $\frac{M}{A}$ is $\oplus$-μ-supplemented.

Corollary4.2. Let M be a $\oplus$-μ-supplemented duo module. Then every factor module of M is a $\oplus$-μ-supplemented module.

Theorem4.3. Let M be a module such that M = $M_1 \oplus M_2$ is a direct sum of submodules M_1 and M_2. Then M_2 is a

$\oplus$-μ-supplemented module if and only if there exists a direct summand B of M such that $B \le M_2$, $M = A + B$ and $A \cap B \ll_\mu B$, for every submodule $\frac{A}{M1}$ of $\frac{M}{M1}$

Proof. ($\Rightarrow$) Let $\frac{A}{M1}$ be any submodule of $\frac{M}{M1}$.Since $A \cap M_2 \le M_2$ and M_2 is $\oplus$-μ-supplemented , then $A \cap M_2$ has μ-supplement say B in M_2 , where $B \oplus B' = M_2$, $M_2 = (A \cap M_2)+B$ and $A \cap M_2 \cap B = A \cap B \ll_\mu B$. Clearly , B is a direct summand of M and $M = M_1 + M_2 = M_1 + (A \cap M_2)+B \le M_1+A+B$, but $M_1 \le A$, then $M = A+B$. So we get the result.

($\Leftarrow$)

Let A be a submodule of M_2 , consider the submodule $\frac{A \oplus M1}{M1}$ of $\frac{M}{M1}$. By our assumption there exists a direct summand B of M such that $B \le M_2$, $M = (A+M_1)+B$ and $(A+M_1) \cap B \ll_\mu B$. Since $M_2 = M_2 \cap M = M_2 \cap [(A+M_1)+B] = B+[(A+M_1) \cap M_2] = B+A+(M_1 \cap M_2) = B+A$,by modular law and since $A \cap B \le (A+M_1) \cap B \ll_\mu B$, then B is μ-supplement of A in M_2. Thus M_2 is a $\oplus$-μ-supplemented.

Proposition4.4. Let M be a $\oplus$-μ-supplemented module. Then $M = M_1 \oplus M_2$, such that $Z^*(M_1) \ll_\mu M_1$ and $Z^*(M_2) = M_2$.

Proof. Since $Z^*(M)$ is a submodule of M and M is $\oplus$-μ-supplemented module , then there exists M_1 such that $M = M_1 \oplus M_2$, for some submodule M_2 of M , $M = Z^*(M)+M_1$ and $Z^*(M) \cap M_1 \ll_\mu M_1$, hence $Z^*(M_1) \ll_\mu M_1$. Since $Z^*(M)$ $Z^*(M_1) \oplus Z^*(M_2)$, then $M = Z^*(M_1) \oplus Z^*(M_2) + M_1 = Z^*(M_2) \oplus M_1$. But $Z^*(M_2) \le M_2$, therefore $Z^*(M_2) = M_2$. Thus , we get the result.

Theorem4.4. For a module M with (D3) the following statements are equivalent.

(1) Every direct summand of M is $\oplus$-μ-supplemented

(2) M is a $\oplus$-μ-supplemented.

(3) $M = M_1 \oplus M_2$, where M_1 is $\oplus$-μ-supplemented with $Z^*(M_1) \ll_\mu M_1$ and M_2 is $\oplus$-μ-supplemented with $Z^*(M_2) = M_2$.

Proof. (1)$\Rightarrow$(2) Clear by the definition.

(2)$\Rightarrow$(1) Proposition (4.3)

(2)$\Rightarrow$(3) Assume that M is a $\oplus$-μ-supplemented with (D3) ,then $M = M_1 \oplus M_2$, where $Z^*(M_1) \ll_\mu M_1$ and $Z^*(M_2) = M_2$,by proposition (4.11) and M_1 , M_2 are $\oplus$-μ-supplemented , by proposition (4.3).

(3)$\Rightarrow$(2) Theorem (4.5).

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