TREATMENT OF SINGULIRITY USING A POST PROCESSING TECHNIQUE APPLIED TO THE SOLUTION OF AN ELLIPTIC BOUNDARY VALUE PROBLEM

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Abstract: *In this paper, a post processing technique is developed to improve the accuracy of the approximation around the corner of an elliptic boundary by determining the coefficients in the known locally convergent expansion. It is used to remove the singularity present at the reentrant corners of the T-shaped region. The reentrant corners singularities which are treated by a post processing technique also makes use of the asymptotic behavior of the solution at the singular points.*

Keywords: post processing technique, locally convergent singularities, reentrant corners, asymptotic behavior.

1. INTRODUCTION

The difficulties of obtaining accurate numerical solutions to Poisson's problems containing boundary singularities are well known. Many of the conventional methods e.g. finite difference, finite element and boundary integral equation methods have limitations when the solution has the singularity on the boundary which is due to either a reentrant corner or to an abrupt change of boundary condition. Many numerical techniques have been proposed to treat these difficulties [5].

In 1988, spectral methods have solved many engineering problems with adequate accuracy, especially in areas of computational fluid dynamics [2]. However, if one wishes to achieve greater accuracy when the exact solution possesses a mild singularity, the use of global trial functions exclusively is inefficient [7] explored the possibility of achieving greater accuracy by using the method of matched Eigen-function [8] expansion to solve the Poisson's equation in a contraction region. They also described how the approximation may be post-processed in the neighbourhood of a reentrant corner singularity in order to obtain an improved and more rapidly converging representation. It is necessary to post-process the approximation since, near the singularity, the expansions converge prohibitively slowly, and one may require a large number of terms to achieve reasonable accuracy. From the computational point of view, it is clearly not efficient to work with expansions containing so many terms. The post-processing technique is performed by matching the known asymptotic form of the singularity to the solution obtained by the method of matched Eigen-function expansions at a sufficient distance from the singularity. The method works because in elliptic problems the effect of the singularity does not penetrate into the interior of the region [3] and so away from the singularity, the Eigen functions converge rapidly. So in principle, we continue this line of investigation and apply the same technique to incorporate the known asymptotic form of the singularity into the numerical technique for solving Poisson's equation in rectangular decomposable domains.

Many numerical techniques have been proposed for the treatment of problems with singularities. Although solutions of elliptic boundary value problems with analytic coefficients and boundary data are analytic where the boundary is analytic, such solutions generally have singularities at corners. When solving Poisson's equation for the T-geometry [1], we have seen that the solutions have singular derivatives at the reentrant corners. The series expansions we obtained in regions I, II, III and IV in the T–geometry converge linearly near the singularity [1]. Away from the singularity a relatively small number of terms are required to achieve good accuracy while near the singularity these expansions converge prohibitively slowly and one may require a large number of tents to achieve a reasonable accuracy. Clearly this is not efficient from the computational point of view. In such problems, high accuracy cannot be obtained by using smooth Eigen functions only. These functions must be augmented with singular functions. So in this paper, we propose a technique which circumvents the slow convergence arising from the behavior of the singularity for the T-geometry. This technique requires knowledge of the local asymptotic form of the singularity [4].

2. MATERIAL AND METHODS

Suppose that there is a boundary singularity and assume that it occurs at the origin. [5] shows that the solution of Poisson's equation can be represented in terms of an expansion of the form

$$U(x,y)=W(x,y)+dx(r,\theta)v(r,\theta) \qquad (1)$$

where $W \varepsilon H^2(\Omega)$, $v(r,\theta)$ is a vertex singular function not in $H^2(\Omega)$ and *x* is a cut-off function. The inclusion of the cut-off function localizes the influence of the singular function to the neighbourhood of the vertex. The singular function is found by solving a harmonic problem in a region sufficiently close to the origin. If an interior angle of ω is subtended at the origin by $\partial\Omega$, where $\partial\Omega$ is the boundary of the domain $\Omega, v(r,\theta)$ takes the form

$$v(r,\theta) = r^{\pi/\omega}\sin(\pi\theta/\omega) \qquad (2)$$

The coefficient *d* in (1), is known as the stress intensity factor, is a continuous linear functional of the source term *f*(*x*,*y*) in Poisson's equation. This is a local functional if and only if π/ω is integral [5]. Thus if π/ω is not integral then the functional is global i.e. it depends on the boundary data in the far field. In our case $\pi/\omega = 2/3$ which means that the functional is global.

Lehman shows that in the neighbourhood of a reentrant corner

$$U(x,y) = \sum_{K=1}^{\infty} d_x v_k(r,\theta) + \text{Inharmonic terms} \quad (3)$$

where the functions v_k are harmonic. We note that this formal power series coincides with (1) where $d = d_1,\ v = v_1$ and

$$W(x,y) = \sum_{k=2}^{\infty} d_x v_k(r,\theta)$$

$$W(x,y) = \sum_{k=1}^{\infty} d_x v_k(r,\theta) + \text{Inharmonic terms} \quad (4)$$

There are numerous techniques available for determining d in (1). Since the Eigen function expansions derived in [1] are highly accurate away from the singularity, the Dirichlet data for U obtained from these functions on $r = R$ can be used to determine the coefficients in the singular series by means of matching.

In this paper we extend the post-processing method of [7]. The key feature of this method is the construction of the inharmonic part of the solution analytically with a high rate of convergence even when the source term is not smooth. This is due to Lehman's discovery that the singular behavior induced by the reentrant corner is confined to harmonic parts of the solution.

3. T-SHAPED GEOMETRY AND BOUNDARY CONDITIONS

We consider the Poisson's equation

$$\nabla^2 U(x,y) = f(x,y)$$

in T-geometry. The domain Ω of the problem and the associated Dirichlet boundary conditions are shown in the Figure 1. We divided the domain Ω into four sub-regions i.e. region I, II, III and IV respectively. Here D is the width of bottom strip.

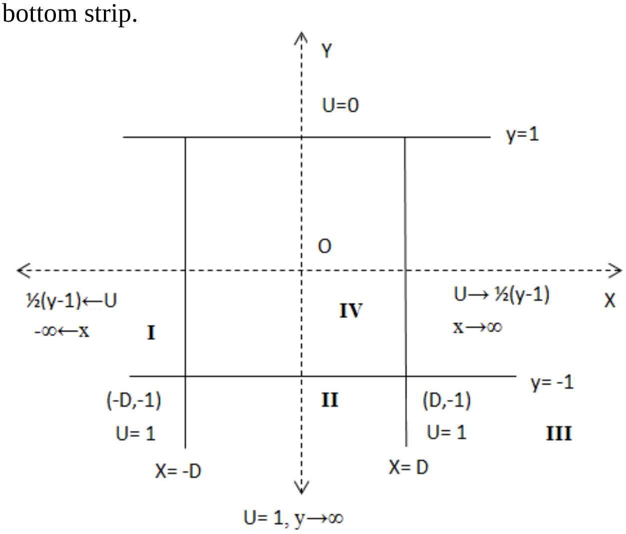

Figure-1 T-geometry and Boundary Conditions.

3.1 Analytical form of the Singularity

We define a sector S around the reentrant corner for the T-geometry in the form

$$S = \{(r,\theta) : 0 \le r \le R,\ 0 \le \theta \le 3\pi/2\} \quad (5)$$

where r, θ are local polar coordinates centered on the reentrant corner. In this sector we consider the Poisson problem

$$U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta} = f(r,\theta) \quad (6)$$

$$U(r,\theta) = U(r,3\pi/2) = 1$$

Assume that the solution of (6) is of the form

$$U(r,\theta) = 1 + \sum_{n=1}^{\infty} d_n(r) \sin\left(\frac{2}{3}\pi\theta\right) \quad (7)$$

Substituting (7) into the differential equation (6) we have

$$\sum_{n=1}^{\infty} \left\{ d_n''(r) + \frac{1}{r} d_n'(r) = \frac{4n^2}{9r^2} d_n(r) \right\} \sin\left(\frac{3}{2}n\theta\right) = f(r,\theta) \text{M}$$

ultiplying both sides of this by $\sin\left(\frac{3}{2}n\theta\right)$, integrating from 0 to $3\pi/2$ and using the orthogonality properties of $\sin\left(\frac{3}{2}n\theta\right)$ which are given

$$\int_0^{\frac{3\pi}{2}} sin\left(\frac{3m\theta}{2}\right) sin\left(\frac{3n\theta}{2}\right) d\theta = \begin{cases} 1 & if\ n = m \\ 0 & if\ n \ne m \end{cases}$$

we obtain

$$d_n''(r) + \frac{1}{r} d_n'(r) = \frac{4n^2}{9r^2} d_n(r) = f_n(r) \quad (8)$$

with $d_n(0) = 0$ and

$$f_n(r) = \frac{4}{3\pi} \int_0^{3\pi/2} f(r,\theta) \sin\left(\frac{2}{3}n\theta\right) d\theta)$$

for all values of n. Using the method of variation of parameters we may write the solution of (8) in the closed form

$$d_n(r) = A_n(\varepsilon) r^{\frac{2n}{3}} + \frac{3}{4n} r^{\frac{2n}{3}} \int_\varepsilon^r t^{(1-\frac{2n}{3})} f_n(t)\,dt$$

$$- \frac{3}{4n} r^{\frac{2n}{3}} \int_\varepsilon^r t^{(1+\frac{2n}{3})} f_n(t)\,dt$$

Provided $f_n(r)$ is regular, it is not difficult to show that the representation in (9) is finite for all $r \ge 0$ and ε is strictly positive.

3.2 Treatment of the singularity for the T-geometry

In the solution of Poisson equation, in the T-geometry slow convergence near the singularity is due to the harmonic parts of the series expansions given by [1] for U^I, U^{II}, U^{III} and

U^{IV} respectively. These harmonic parts are given by:

$$U^{I} = \frac{1}{2}(1-y) + \sum_{n=1}^{N} b_n e^{\frac{n\pi}{2}(x+D)} \sin\left\{\frac{n\pi}{2}(1+y)\right\} \quad (10)$$

$$U^{II} = 1 + \sum_{n=1}^{N} a_n e^{\frac{n\pi}{2D}(1+y)} \sin\left\{\frac{n\pi}{2D}(D+x)\right\} \quad (11)$$

$$U^{III} = \frac{1}{2}(1-y) + \sum_{n=1}^{N} b_n e^{\frac{n\pi}{2}(x+D)} \sin\left\{\frac{n\pi}{2}(1+y)\right\} \quad (12)$$

and

$$U^{IV} = \frac{1}{2}(1-y) + \sum_{n=1}^{N} \frac{\cosh\left(\frac{n\pi x}{2}\right)}{\cosh\left(\frac{n\pi D}{2}\right)} b_n \sin\left\{\frac{n\pi}{2}(1+y)\right\} + \sum_{n=1}^{N} \frac{\sinh\left(\frac{n\pi}{2}(1-y)\right)}{\sinh\left(\frac{n\pi x}{D}\right)} b_n \sin\left\{\frac{n\pi}{2D}(D+x)\right\} \quad (13)$$

respectively. The coefficients a_n and b_n have been determined in [1]. We note as before that (10) to (13) do not form the full harmonic parts of U^I, U^{II}, U^{III} and U^{IV}, respectively, in [1]. To construct the harmonic parts we proceed in a way similar to that of the contraction geometry [7] and denote them by $U_h^I, U_h^{II}, U_h^{III}$ and U_h^{IV} respectively.

We consider the reentrant corner at $(-D,-1)$ of the T-geometry [1] and define a sector S around it by (5), where

$$r^2 = (x+D)^2 + (1+y)^2, \quad \theta = \tan^{-1}\left[-\frac{1+y}{D+x}\right] \quad (14)$$

We may write the approximate solution valid in sector S of the form

$$U_s(r,\theta) = 1 + \sum_{l=1}^{\infty} d_1 r^{\frac{2}{3}l} \sin\left(\frac{2}{3}l\theta\right) \quad (15)$$

Using the same post-processing technique as before we use the matching process which requires the following

$$U_s(R,\theta) = \begin{cases} U_h^I[x_R(\theta), y_R(\theta)], & 0 \le \theta \le \frac{\pi}{2}, \\ U_h^{IV}[x_R(\theta), y_R(\theta)], & \frac{\pi}{2} \le \theta \le \pi, \\ U_h^{II}[x_R(\theta), y_R(\theta)], & \pi \le \theta \le \frac{3}{2}\pi, \end{cases} \quad (16)$$

where $X_R = -D - R\cos\theta, y_R = -1 + R\sin\theta$. Multiplying both sides of (16) by $\sin\left(\frac{2}{3}k\theta\right)$, integrating from 0 to $\frac{3}{2}\pi$ and using the orthogonality conditions we obtain

$$d_k = \left\{\frac{3\pi}{4} r^{\frac{2k}{3}}\right\}^{-1} \left[\int_0^{\pi/2} U_h^I \sin\left(\frac{2}{3}k\theta\right)d\theta + \int_{\pi/2}^{\pi} U_h^{IV} \sin\left(\frac{2}{3}k\theta\right)d\theta + \int_{\pi/2}^{3\pi/2} U_h^{II} \sin\left(\frac{2}{3}k\theta\right)d\theta = e_k\right] \quad (17)$$

where

$$e_k = \begin{cases} \frac{3}{k}, & k \ \text{is odd.} \\ 0, & k \ \text{is even} \end{cases} \quad (18)$$

The integral in the right-hand side of (17) are computed by Filon's method. It is only the harmonic parts which are replaced by the expansion (16) in the sector. We may compute the remaining parts as accurately as we wish.

4. NUMERICAL RESULTS

We present the numerical result obtained using the post processing technique for the T-geometry [6].

4.1 Laplace Equation

Consider Laplace's equation for the T-geometry the domain Ω and the boundary conditions are shown in the Figure 1. The series expansions for U^I, U^{II}, U^{III} and U^{IV} given by [1] converge slowly in the sector S defined by (5) and (14) and are replaced by an expansion (15) which holds in the sector S where the coefficients d_k are obtained by (17). We list the first four coefficients d_k in (15) for different values of R computed with a value of $M = 33$ in table (2). The coefficients of d_k seem to become independent of R and tend to some finite value as R increase to unity which is expected as the matched Eigen functions expansion converges rapidly away from the singularity. The largest value of R, we take is unity. Table (1) in which case the radius of the sector is tangent to the line of symmetry of the geometry (i.e. x=0). Only eight non-vanishing terms are needed.

Table-1 The coefficients d_n for different values of R in the T-Geometry

R	d_1	d_2	d_3	d_4
0.1250	–0.42992	–0.18282	0.00120	0.00879
0.2500	–0.43061	–0.18282	0.00120	0.00851
0.375-	–0.43079	–0.18282	0.00120	0.00850
0.5000	–0.43086	–0.18282	0.00120	0.00849
0.6250	–0.4309	–0.18282	0.00120	0.00849
0.7500	–0.43093	–0.18282	0.00120	0.00849
0.8750	–0.43094	–0.18282	0.00120	0.00849
1.0000	–0.43096	–0.18282	0.00120	0.00849

Again to make the choice of $M = 33$ justifiable we consider the convergence of the coefficients d_k = 1, … , 4 for $R = 1$ and evaluate them for various values of M. Table (2) shows that the coefficients d_k do not change when M varies after 33 to 97, which indicates that choice of $M = 33$ is an adequate number.

Table-2 The rate of convergence of d_n for different values of M in the T-Geometry

M	d_1	d_2	d_3	d_4
17	–0.4309	–0.18282	0.00120	0.00849
33	–0.43096	–0.18282	0.00120	0.00849
49	–0. 43096	–0.18282	0.00120	0.00849
65	–0. 43096	–0.18282	0.00120	0.00849
81	–0. 43096	–0.18282	0.00120	0.00849
97	–0. 43096	–0.18282	0.00120	0.00849

For Poisson's equation discussed in [1] with different values of α and β, we noticed that the slow convergence near the singularities is due to the harmonic parts of the series solution given by (10) to (13) for U^I, U^{II}, U^{III} and U^{IV} in the sector S defined by (5) and (14). We replaced it by the singular expansions (15) for which the coefficients d_k are obtained from (17). In this case the coefficients a_n and b_n are obtained and are shown in tables for $\alpha = 1, \beta = 0$ and [1]. The singular coefficients are almost identical to those obtained for harmonic problem and therefore we do not tabulate them. The regular part may be calculated as accurately as we wish. A contour plot of the harmonic problem in the T-geometry is shown in the Figure (1).

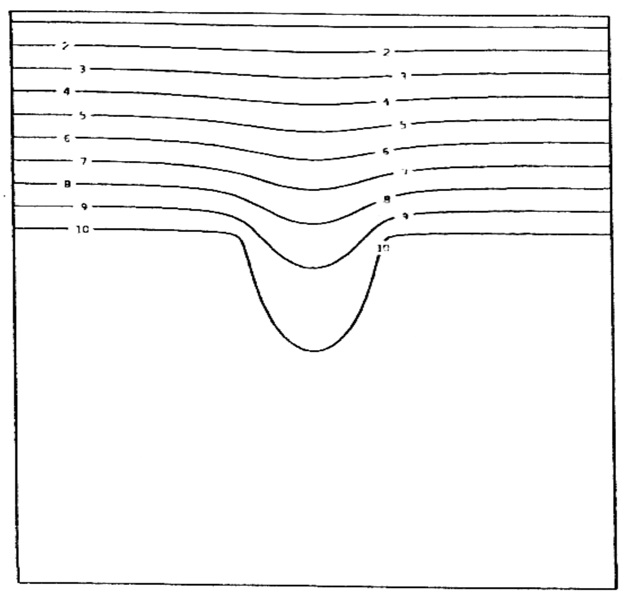

Figure-2 The post-processed contours of the solution of Laplace equation $\nabla^2 U = 0$ in the T-shaped geometry.

5. CONCLUSION

Reentrant corner singularities are treated by a post-processing technique, which makes use of the known asymptotic behaviour of the solution at the singular points. We assess the relative advantages and disadvantages of the various aspects of this method. A post-processing technique is developed to improve the accuracy of the approximation around corners by determining the coefficients in the known locally convergent expansions. This is necessary because of the singularities present at the reentrant corners. The most attractive feature of this technique is how an inharmonic part of the solution can be constructed analytically with a high rate of convergence of series solution even when $f(x,y)$ is not smooth. This is due to Lehman's discovery that singular behaviour induced by the reentrant corners is confined to harmonic parts of the solution.

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