POLYNOMIOGRAPHY VIA GENERALIZED NEWTON-RAPHSON'S METHOD FREE FROM SECOND DERIVATIVE

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ABSTRACT: *The aim of this paper is to present polynomiography using generalized Newton-Raphson's method free from second derivative for finding the roots of a given complex polynomial. This method is extracted from generalized Newton-Raphson's method by removing its second derivative using finite difference scheme. Polynomiography is the art and science of visualization in approximation of zeros of complex polynomials. The images thus obtained are called polynomiographs. In this paper, we obtain polynomiographs of different complex polynomials using generalized Newton-Raphson's method free from second derivative. The obtained polynomiographs have very interesting patterns for complex polynomial equations. We believe that the results of this paper enrich the functionality of the existing polynomiography software.*

2010 MSC subject classification:
Key words: Polynomials, Iterative method, Fractals, Polynomiographs.

INTRODUCTION

Polynomials are one of the most significant objects in many fields of mathematics. Polynomial root-finding has played a key role in the history of mathematics. It is one of the oldest and most deeply studied mathematical problems. The last interesting contribution to the polynomials root finding history was made by Kalantari [16,17], who introduced the polynomiography. As a method which generates nice looking graphics, it was patented by Kalantari in USA in 2005 [17,18]. Polynomiography is defined to be "the art and science of visualization in approximation of the zeros of complex polynomials, via fractal and non fractal images created using the mathematical convergence properties of iteration functions" [16]. An individual image is called a "polynomiograph". Polynomiography combines both art and science aspects. Polynomiography gives a new way to solve the ancient problem by using new algorithms and computer technology. Polynomiography is based on the use of one or an infinite number of iterative methods formulated for the purpose of approximation of the root of polynomials e.g. Newton's method, Halley's method, Householder's method etc. The word "fractal", which partially appeared in the definition of polynomiography, was coined by the famous mathematician Benoit Mandelbrot [15]. Both fractal images and polynomiographs can be obtained via different iterative schemes. Fractals are self-similar has typical structure and independent of scale. On the other hand, polynomiographs are quite different. The "polynomiographer" can control the shape and designed in a more predictable way by using different iterative methods to the infinite variety of complex polynomials. Generally, fractals and polynomiographs belong to different classes of graphical objects.

Polynomiography has diverse applications in mathematics, science, education, art and design. According to Fundamental Theorem of Algebra, any complex polynomial with complex coefficients $\{a_n, a_{n-1}, \dots, a_1, a_0\}$

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0 \quad (1)$$

of degree n has n roots (zeros) which may or may not be distinct. The degree of polynomial describes the number of basins of attraction and placing roots on the complex plane manually localization of basins can be controlled.

Usually, polynomiographs are colored based on the number of iterations needed to obtain the approximation of some polynomial root with a given accuracy and a chosen iteration method. The description of polynomiography, its theoretical background and artistic applications are described in [16,17,18].

ITERATION

During the last century, various numerical techniques for solving nonlinear equation $f(x)=0$ have been successfully applied. For examples see [1-8, 12-14], and the reference therein. Now we define:

For a given x_0, compute the approximate solution x_{n+1} by the following iterative schemes:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = y_n - \frac{f(y_n)f'(y_n) - \sqrt{f^2(x_n)f'^2(y_n) - 2f(x_n)f(y_n)f'(x_n)[f'(x_n) - f'(y_n)]}}{f'(x_n)[f'(x_n) - f'(y_n)]}$$

This is so-called generalized Newton-Raphson's method free from second derivative for solving nonlinear equations [1]. Let $p(z)$ be the complex polynomial, then

$$y_n = z_n - \frac{p(z_n)}{p'(z_n)}$$

$$x_{n+1} = y_n - \frac{p(y_n)p'(y_n) - \sqrt{p^2(z_n)p'^2(y_n) - 2p(z_n)p(y_n)p'(z_n)[p'(z_n) - p'(y_n)]}}{p'(z_n)[p'(z_n) - p'(y_n)]}$$

where $z_o \in \mathbb{C}$ is a starting point, is so-called generalized Newton-Raphson's method free from second derivative for solving nonlinear complex polynomials. The sequence $\{z_n\}_{n=0}^{\infty}$ is called the orbit of the point z_o converges to a root z^* of p then, we say that z_o is attracted to z^*. A set of all such starting points for which $\{z_n\}_{n=0}^{\infty}$ converges to root z^* is called the basin of attraction of z^*.

APPLICATIONS

The applications of the generalized Newton-Raphson's method free from second derivative for solving nonlinear complex equations perturbs the shape of polynomial basins and makes the polynomiographs look more "fractal". The aim of using the generalized Newton-Raphson's method free from second derivative for solving nonlinear complex equations is to create images that are quite new, different from images by the Newton's method and Householder's method free from second derivatives [2] and [9,10,11], and interesting from the aesthetic point of view.

In this section we present some examples of polynomiographs for different complex polynomials equation $p(z)=0$. The different colors of images depend upon number of iterations to reach a root with given accuracy $\varepsilon = 0.001$. One can obtain infinitely many nice looking polynomiographs by changing parameter k, where k is the upper bound of the number of iterations.

In this paper, we set $k = 12$.

Polynomiograph for $z^3 - 1 = 0$

Complex polynomial equation $z^3 - 1 = 0$, having three roots. The polynomiograph is presented in the following figure with three distinct basins of attraction to the three roots of the polynomial $z^3 - 1 = 0$.

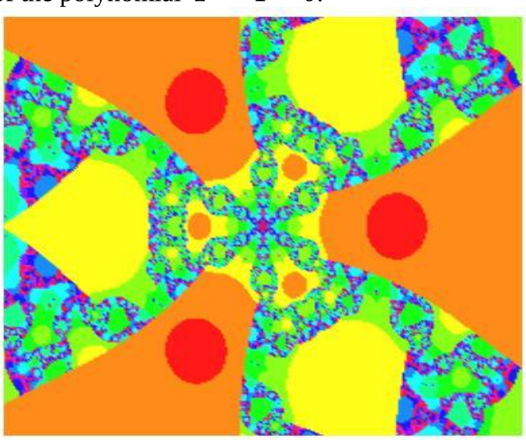

Fig. 1. Polynomiography for $z^3 - 1 = 0$.

Polynomiograph for $z^3 - 3 = 0$

Complex polynomial equation $z^3 - 3 = 0$, having three roots. The polynomiograph is presented in the following figure with three distinct basins of attraction to the three roots of the polynomial $z^3 - 3 = 0$.

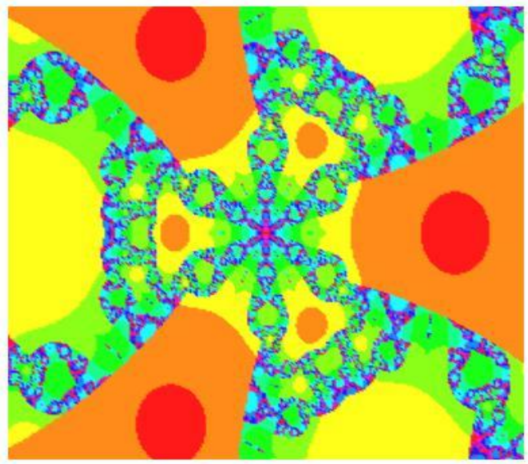

Fig. 2. Polynomiography for $z^3 - 3 = 0$.

Polynomiograph for $z^3 - z + 1 = 0$

Complex polynomial equation $z^3 - z + 1 = 0$, having three roots. The polynomiograph is presented in the following figure with three distinct basins of attraction to the three roots of the polynomial $z^3 - z + 1 = 0$.

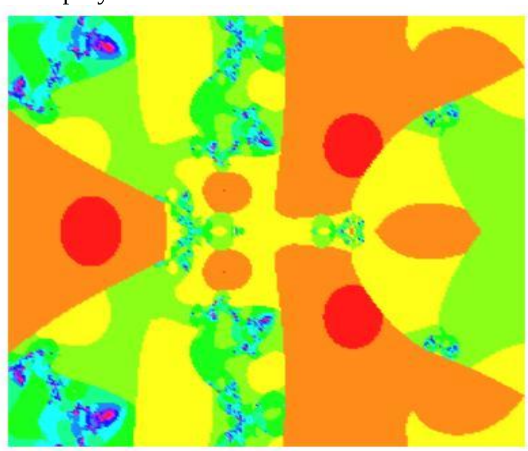

Fig.3. Polynomiography for $z^3 - z + 1 = 0$.

Polynomiograph for $z^3 - z^2 + z - 1 = 0$

Complex polynomial equation $z^3 - z^2 + z - 1 = 0$, having three roots. The polynomiograph is presented in the following figure with three distinct basins of attraction to the three roots of the polynomial $z^3 - z^2 + z - 1 = 0$.

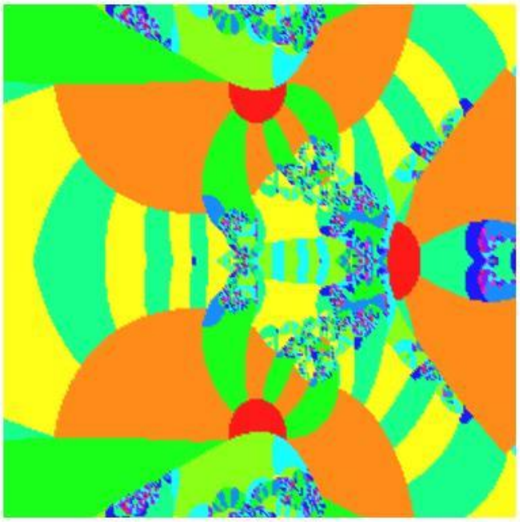

Fig. 4. Polynomiography for $z^3 - z^2 + z - 1 = 0$.

Polynomiograph for $(z + 1)(z^2 + 2) = 0$

Complex polynomial equation $(\mathrm{z} + 1)(z^2 + 2) = 0$, having three roots. The polynomiograph is presented in the following figure with three distinct basins of attraction to the three roots of the polynomial $(\mathrm{z} + 1)(z^2 + 2) = 0$.

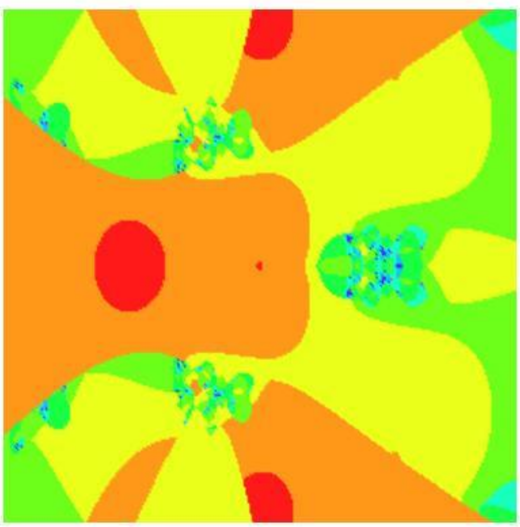

Fig. 5. Polynomiography for $(\mathrm{z} + 1)(z^2 + 2) = 0$.

Polynomiograph for $z^4 - 1 = 0$

Complex polynomial equation $z^4 - 1 = 0$, having four roots. The polynomiograph is presented in the following figure with four distinct basins of attraction to the four roots of the polynomial $z^4 - 1 = 0$.

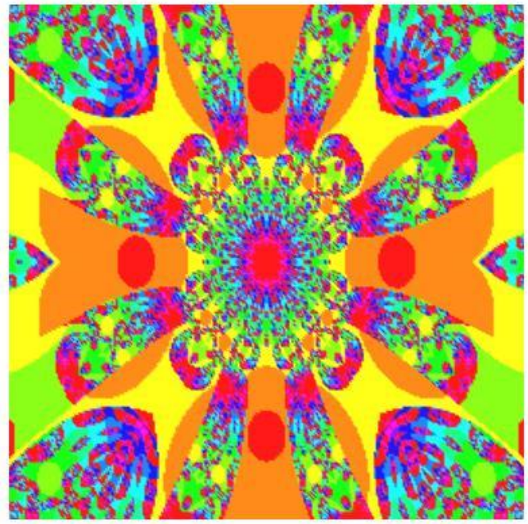

Fig. 6.Polynomiography for $z^4 - 1 = 0$.

Polynomiograph for $z^4 - 4 = 0$

Complex polynomial equation $z^4 - 4 = 0$, having four roots. The polynomiograph is presented in the following figure with four distinct basins of attraction to the four roots of the polynomial $z^4 - 4 = 0$.

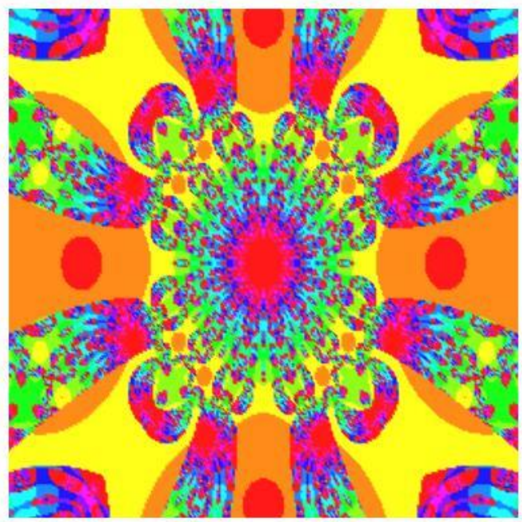

Fig. 7.Polynomiography for $z^4 - 4 = 0$.

Polynomiograph for $\mathbf{z}^4 + \mathbf{z}^3 - 1 = 0$

Complex polynomial equation $z^4 + z^3 + 3 = 0$, having four roots. The polynomiograph is presented in the following figure with four distinct basins of attraction to the four roots of the polynomial $z^4 + z^3 - 1 = 0$.

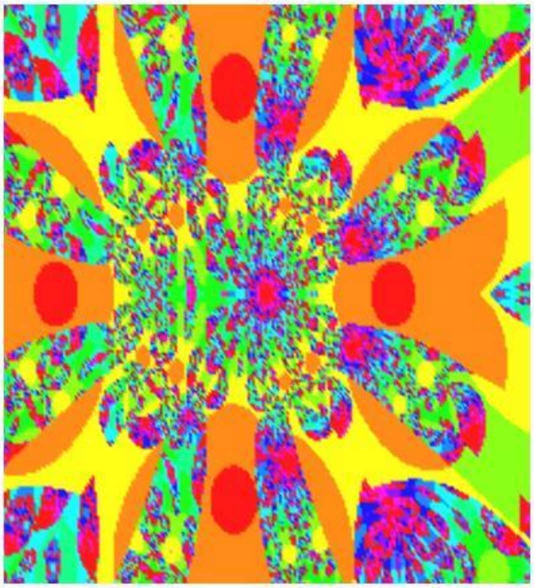

Fig. 8.Polynomiography for $z^4 + z^3 - 1 = 0$.

Polynomiograph for $\mathbf{z^4 + z^2 - 1}$

Complex polynomial equation $z^4 + z^2 - 1 = 0$, having four roots. The polynomiograph is presented in the following figure with four distinct basins of attraction to the four roots of the polynomial $z^4 + z^2 + 1 = 0$.

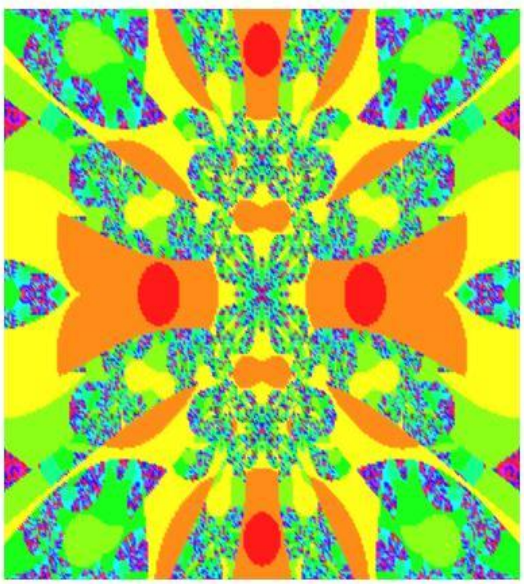

Fig. 9.Polynomiography for $z^4 + z^2 - 1 = 0$.

Polynomiograph for $\boldsymbol{z^4 - 2z + 1}$

Complex polynomial equation $z^4 - 2z + 1 = 0$, having four roots. The polynomiograph is presented in the following figure with four distinct basins of attraction to the four roots of the polynomial $z^4 - 2z + 1 = 0$.

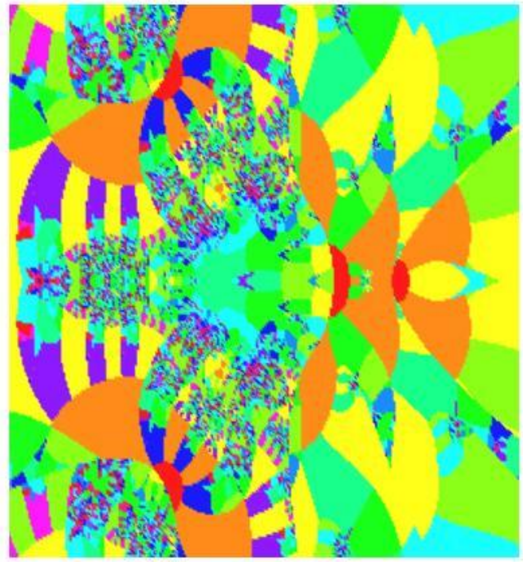

Fig. 10.Polynomiography for $z^4 - 2z + 1 = 0$.

Polynomiograph for $\boldsymbol{(z^2 + 1)(z^2 + 2) = 0}$

Complex polynomial equation $(z^2 + 1)(z^2 + 2) = 0$, having four roots. The polynomiograph is presented in the following figure with four distinct basins of attraction to the four roots of the polynomial $(z^2 + 1)(z^2 + 2) = 0$.

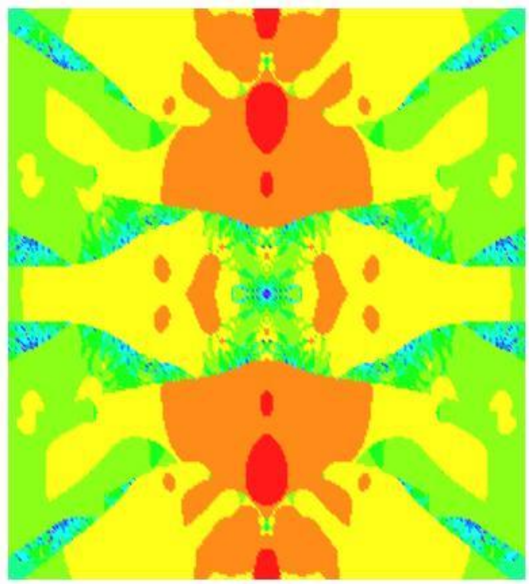

Fig. 11. Polynomiography for $(z^2 + 1)(z^2 + 2) = 0$.

Polynomiograph for $\boldsymbol{(z^2 + 5)(z^2 + 6) = 0}$

Complex polynomial equation $(z^2 + 5)(z^2 + 6) = 0$, having four roots. The polynomiograph is presented in the following figure with four distinct basins of attraction to the four roots of the polynomial $(z^2 + 5)(z^2 + 6) = 0$.

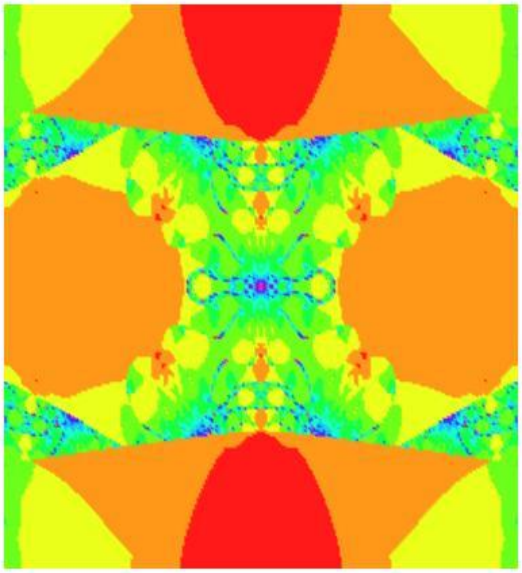

Fig. 12. Polynomiography for $(z^2+5)(z^2+6)=0$.

Polynomiograph for $z^5-1=0$

Complex polynomial equation $z^5-1=0$, having five roots. The polynomiograph is presented in the following figure with five distinct basins of attraction to the five roots of the polynomial $z^5-1=0$.

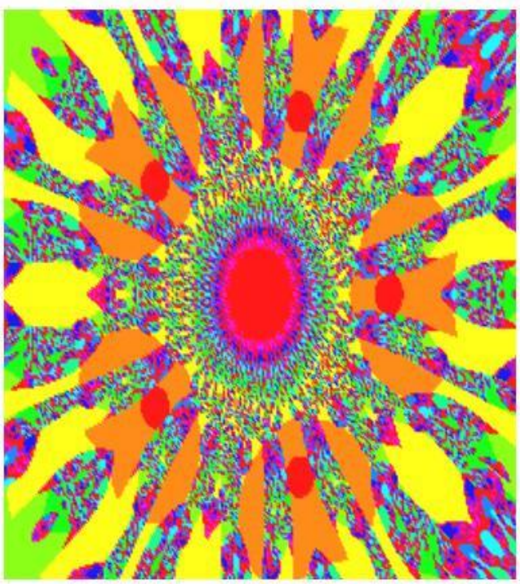

Fig. 13.Polynomiograph for $z^5-1=0$.

Polynomiograph for $z^5-z^4-4=0$

Complex polynomial equation $z^5-z^4-4=0$, having five roots. The polynomiograph is presented in the following figure with five distinct basins of attraction to the five roots of the polynomial $z^5-z^4-4=0$.

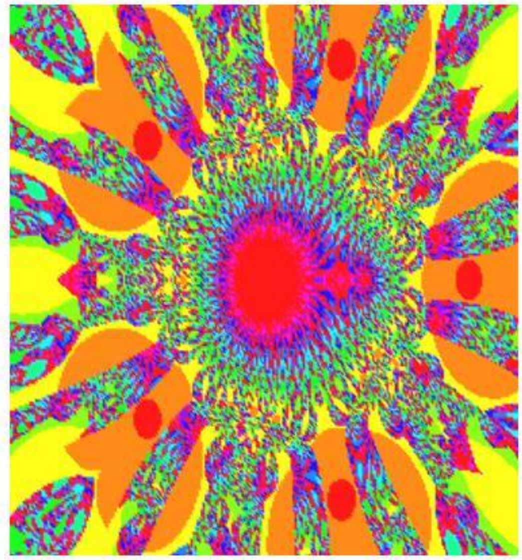

Fig. 14.Polynomiograph for $z^5-z^4-4=0$.

Polynomiograph for $z^5-z^3+3=0$

Complex polynomial equation $z^5-z^3+3=0$, having five roots. The polynomiograph is presented in the following figure with five distinct basins of attraction to the five roots of the polynomial $z^5-z^3+3=0$.

Fig. 15.Polynomiograph for $z^5-z^3+3=0$.

Polynomiograph for $z^5+z+5=0$

Complex polynomial equation $z^5+z+5=0$, having five roots. The polynomiograph is presented in the following figure with five distinct basins of attraction to the five roots of the polynomial $z^5+z+5=0$.

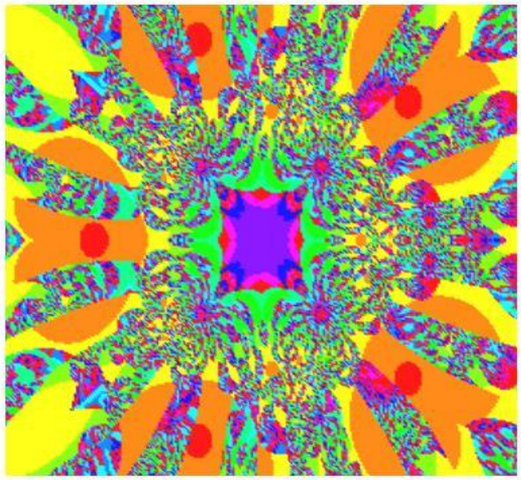

Fig. 16.Polynomiograph for $z^5 + z + 5 = 0$.

Polynomiograph for $z(z^2 + 8)(z^2 + 9) = 0$

Complex polynomial equation $z(z^2 + 8)(z^2 + 9) = 0$, having five roots. The polynomiograph is presented in the following figure with five distinct basins of attraction to the five roots of the polynomial $z(z^2 + 8)(z^2 + 9) = 0$.

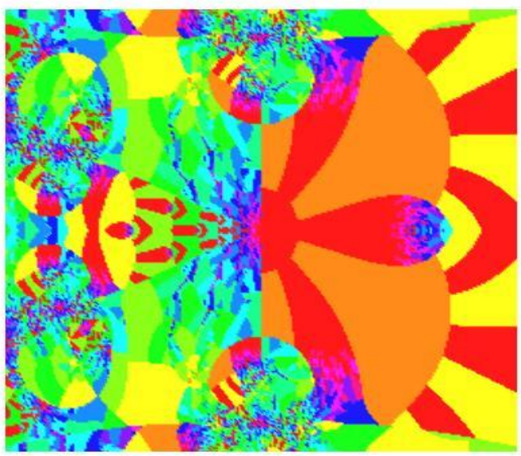

Fig. 17.Polynomiograph for $z(z^2 + 8)(z^2 + 9) = 0$.

C0NCLUSIONS

We present some examples of polynomiographs for different complex polynomial equations $p(z) = 0$. We used generalized Newton-Raphson's method free from second derivative for solving nonlinear complex polynomial equations to create images that are quite new, different from images by the Newton's method and Householder's method free from second derivatives [2], and interesting from the aesthetic point of view.

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