EXPLORING THE METRIC DIMENSION OF HYPERCUBES: AN IN-DEPTH ANALYSIS

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ABSTRACT: *For an ordered set $W = \{w_1, w_2, \dots, w_n\}$ of vertices and a vertex v in a connected graph G, the representation of v with respect to W is the ordered k-tuple $r(v \mid W) = \{d(v, w_1), d(v, w_2), \dots, d(v, w_n)\}$. The set W is called a resolving set for G if every two vertices of G have distinct representations. A resolving set containing a minimum number of vertices is called a basis for G. The dimension of G, denoted by β(G) is the number of vertices in a basis of G. These studies expose and elucidate using the concepts in linear algebra to solve for the metric dimension of hypercube in the article of A. F. Beardon entitled, "Resolving the Hypercube." Many results have been established. In particular, the following results have been expose and elucidated: the metric dimension of the n-dimensional hypercube is less than or equal to n, the metric dimension of n-dimensional hypercube is less than or equal to the metric dimension of (n+1)-dimensional hypercube, and the metric dimension of the (m+n)- dimensional hypercube is less than or equal to the metric dimension of the m-dimensional hypercube plus the metric dimension of n-dimensional hypercube. The extension of results are as follows: the metric dimension of a 10-dimensional hypercube is equal to 7, the metric dimension of n-dimensional hypercube is less than or equal to n – 3, the metric dimension of a 12-dimensional hypercube is equal to 8 and the metric dimension of n-dimensional hypercube is less than or equal to n – 4. This is an expository article with extensions that uses the techniques in linear algebra to solve for the metric dimension of n-dimensional hypercubes. Our main result is to identify that the set of vertices that resolves a hypercube.*

Keywords: metric dimension, resolving sets, *n*-dimensional hypercube Q^n

1. INTRODUCTION.

The concept of metric dimension of a graph was defined as early as 1950's but was named then location number instead of metric dimension. The term metric dimension, which was used widely rather than location number, was first introduced by Harary and Melter in 1970's. Since 1975 much has been published about the metric dimension of graphs. These studies focused on the problem of estimating the metric dimension of hypercube.

Objectives of the Study

The researcher aims to expose the results in the article entitled "Resolving the Hypercube" by A.F. Beardon. To achieve this end, the researcher attempted to elucidate the discussion of the following results and their extensions:

1. The metric dimension of the *n*-dimensional hypercube is less than or equal to *n*.
2. The metric dimension of *n*-dimensional hypercube is less than or equal to the metric dimension of (*n+1*)-dimensional hypercube.
3. The metric dimension of the (*m*+*n*)-dimensional hypercube is less than or equal to the metric dimension of the *m*-dimensional hypercube plus the metric dimension of *n*-dimensional hypercube.
4. To provide a thorough discussion of the different concepts used in this paper.

Significance of the Study

In their study, Caseres et al. (2007) pointed out that resolving sets arise in many diverse areas including coin weighting, network discovery and verification, robot navigation and strategies for the Mastermind game.

Furthermore, the results of this study will go a long way towards a more complete understanding of the metric dimension of graphs. In the future, this study may serve as a link between the present information on metric dimension of graphs and future investigation on the subject.

Moreover, the results generated in this study are important as they may lead to further related studies and trigger more substantial results on this topic.

Scope and Limitations

Basically, this study deals mainly on metric dimension in graphs. Focusing on finding the metric dimension of hypercubes. To attain the objectives, we use the concepts similar to those found in algebra.

Review of Related Literature and Studies

The concept, Metric Dimension in Graphs was introduced in1953 by Blumenthal. However, from 1953 to 1974, it did not grab much attention. In 1975, Chartrand (2000) presented the problem on estimating the value of β_n. The problem seemed to have spur up the interest on the concept, thus, much has been published about the metric dimension of graphs, for example Bailey (2010), Caseres et al. (2007), Chartrand (2000) and Harary (1976).

So far, the known values of β_n are

$$\beta_n = \begin{cases} n & \text{, if } n = 1,2,3,4, \\ n-1 & \text{, if } n = 5,6,7 \\ n-2 & \text{, if } n = 8,9 \\ n-3 & \text{, if } n = 10,11 \\ n-4 & \text{, if } n = 12,13 \\ n-5 & \text{, if } n = 14,15,16 \\ n-6 & \text{, if } n = 17 \end{cases}$$

(Caseres et al., 2002; Kratica et al., 2009). Most of these results are verified using computer search.

The metric dimension of the hypercube β_n, satisfy the inequalities $\beta_n \le n$, $\beta_n \le \beta_{n+1}$, and $\beta_{m+n} \le \beta_m + \beta_n$. (Lindström, 1964). He also proved that; $n\log 2/\log(n+1)$ is a lower bound for β_n. Deeper probabilistic arguments would show that $\beta_n \sim n\log 4/\log n$ as n increases without a bound. (Lindström, 1965).

Beardon (2013) used techniques from linear algebra to verify the inequalities suggested by Lindstrom (1964). Beardon (2013) reduced the question of whether a set of vertices resolves the hypercube to the question of whether or not a set of linear equations has a non-trivial solution.

Earlier, Harary et al. (1988) proved that every hypercube graph is bipartite. The two colors of this coloring may be found from the subset construction of hypercube graphs, by giving one color to the subsets that have an even number of elements and the other color to the subsets with an odd number of elements.

Moreover, they proved that every hypercube graph is Hamiltonian and every hypercube Q^n with n > 1 has a Hamiltonian cycle which is a cycle that visits each vertex exactly once. Additionally, a Hamiltonian path exists between two vertices *u* and *v* if and only if they have different colors in a 2-coloring of the graph. This is called, Hamiltonicity.

The Hamiltonicity of the hypercube is tightly related to the theory of Gray codes. More precisely, it days that there is a bijective correspondence between the set of *n*-bit cyclic Gray codes and the set of Hamiltonian cycles in the hypercube Q^n. An analogous property holds for acyclic *n*-bit Gray codes and Hamiltonian paths. A lesser known fact is that every perfect matching in the hypercube extends to a Hamiltonian cycle. The question whether every matching extends to a Hamiltonian cycle remains an open problem. (Rusky et al., 2007)

Other properties of the hypercube are as follows(Harary et al., 1988). The hypercube graph Q^n is the Hasse diagram of a finite Boolean algebra; is a median graph (every median graph is an isometric subgraph of a hypercube, and can be formed as a retraction of a hypercube); has more than 22*n* – 2 perfect matchings (this is another consequence that follows easily from the inductive construction); is arc transitive and symmetric; is an *n*-vertex-connected graph; by Balinski's theorem, is planar (can be drawn with no crossings) if and only if n ≤ 3. For larger values of *n*, the hypercube has genus and has exactly spanning trees. The family Qn (n > 1) is a Levy family of graphs.

The achromatic number of Q^n is known to be proportional to $\sqrt{n2^n}$, but the constant of proportionality is not known precisely. The bandwidth of Q^n is exactly

$$\sum_{i=0}^{n}\binom{n}{\lfloor n/2 \rfloor}.$$ (Harper, 2001)

The eigenvalues of the adjacency matrix of a hypercube are $-n,-n+2,-n+4,\ldots,n-4,n-2,n$ and the eigenvalues of its Laplacian are $0,2,\ldots,2n$. The *k*-th eigenvalue has multiplicity in both cases. The isoperimetric number is $h(G)=1$. (Harary et al., 1988).

The following are open problems. The problem of finding the longest path or cycle that is an induced subgraph of a given hypercube graph is known as the *snake in the box problem*. Szymanski's conjecture concerns the suitability of a hypercube as an network topology for communications. It states that, no matter how one chooses a permutation connecting each hypercube vertex to another vertex with which it should be connected, there is always a way to connect these pairs of vertices by paths that do not share any directed edge. (Harary et al., 1988)

A graph is an ordered pair *(V, E)* where *V* is a finite non-empty set and *E* is a set of some two element subsets of *V*. For example, *G = ({a, b, c}, {ab, bc})* is a graph.

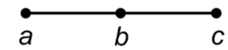

Figure 1. The Graph G

For an ordered set $W=\{w_1,w_2,\ldots,w_n\}$ of vertices and a vertex *v* in a connected graph *G*, the representation of *v* with respect to *W* is the ordered *k*-tuple $r(v \mid W) = \{d(v, w_1), d(v, w_2), \ldots, d(v, w_n)\}$. The set *W* is called a resolving set for *G* if every two vertices of *G* have distinct representations. A resolving set containing a minimum number of vertices is called a basis for *G*. The dimension of *G*, denoted by *β*(*G*) is the number of vertices in a basis of *G*. Consider the metric dimension of graph *J* in figure 2. Let the set $W_1 = \{a, e\}$. Then $r(a \mid W_1)=\langle 0,1\rangle$, $r(b \mid W_1)=\langle 1,1\rangle$, $r(c \mid W_1)=\langle 2,2\rangle$, $r(d \mid W_1)=\langle 2,1\rangle$, and $r(e \mid W_1)=\langle 1,0\rangle$. Hence, W_1 is a resolving set of *J*. Thus, $\beta(J)\le 2$. Then by brute force $\beta(J)\neq 1$. Note that as shown by Table 1, W_i for *i*=2, 3, 4, 5, 6 are not resolving set of *J*. Therefore, W_1 is a minimum resolving set of *J*, that is, W_1 is a basis of *J*. Accordingly, the metric dimension of *J* is equal to 2.

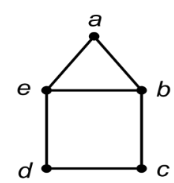

Figure 2: The Graph *J*

Table 1. Representations of a, b, c, d, e with respect to W_2, W_3, W_4, W_5, and W_6.

$W_2=\{a\}$	$W_3=\{b\}$	$W_4=\{c\}$	$W_5=\{d\}$	$W_6=\{e\}$
$r(a \mid W_2)=\langle 0\rangle$	$r(a \mid W_2)=\langle 1\rangle$	$r(a \mid W_2)=\langle 2\rangle$	$r(a \mid W_2)=\langle 2\rangle$	$r(a \mid W_2)=\langle 0\rangle$
$r(b \mid W_2)=\langle 1\rangle$	$r(b \mid W_2)=\langle 1\rangle$	$r(b \mid W_2)=\langle 1\rangle$	$r(b \mid W_2)=\langle 2\rangle$	$r(b \mid W_2)=\langle 1\rangle$
$r(c \mid W_2)=\langle 2\rangle$	$r(c \mid W_2)=\langle 1\rangle$	$r(c \mid W_2)=\langle 0\rangle$	$r(c \mid W_2)=\langle 1\rangle$	$r(c \mid W_2)=\langle 2\rangle$
$r(d \mid W_2)=\langle 2\rangle$	$r(d \mid W_2)=\langle 2\rangle$	$r(d \mid W_2)=\langle 1\rangle$	$r(d \mid W_2)=\langle 0\rangle$	$r(d \mid W_2)=\langle 1\rangle$
$r(e \mid W_2)=\langle 1\rangle$	$r(e \mid W_2)=\langle 1\rangle$	$r(e \mid W_2)=\langle 2\rangle$	$r(e \mid W_2)=\langle 1\rangle$	$r(e \mid W_2)=\langle 0\rangle$

In graph theory, the *n*-dimensional hypercube Q^n is a graph whose vertices are the *n*-dimensional binary vectors, where two vertices are adjacent if they differ in exactly one coordinate.

In this study, the metric dimension of a hypercube is denoted by *β*(*G*). The problem of finding the dimension of a hypercube will be solved using the concepts in linear algebra.

Construction of a Hypercube

The hypercube graph Q^n may be constructed using 2^n vertices labeled with *n*-bit binary numbers and connecting two vertices by an edge whenever the Hamming distance of their labels is 1.

Alternatively, Q^{n+1} may be constructed from the disjoint union of two hypercubes Q^n, by adding an edge from each vertex in one copy of Q^n to the corresponding vertex in the other copy.

Notation: The following notations are used.

$0=(0,0,0,\ldots,0)$; $e_1=(1,0,0,\ldots,0)$; $e_2=(0,1,0,\ldots,0)$; …; $e_n=(0,0,0,\ldots,0,1)$ and $E=(1,1,1,\ldots,1)$.

Consider Q^n for n = 3. We have $Q^3 = (V, E)$, where V ={(0,0,0), (0,0,1), (0,1,1), (1,0,0), (0,1,0), (1,1,0), (1,1,1)} and E = {(0,0,0)(0,0,1), (0,0,0)(0,1,0), (0,0,0)(1,0,0), (0,0,1)(0,1,1), (0,0,1)(1,0,1), (0,1,0)(0,1,1), (0,1,0)(1,1,0), (0,1,1)(1,1,1), (1,0,0)(1,0,1), (1,0,0)(1,1,0), (1,0,1)(1,1,1), (1,1,0)(1,1,1)}. Q^3 is show in figure 3.

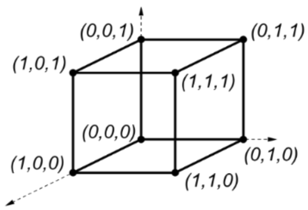

Figure 3. Graph Q^3

2. RESULTS AND DISCUSSIONS

Theorem 2.1. The metric dimension of the n-dimensional hypercube $\beta_n \geq \frac{n\log 2}{\log(n+1)}$.

Proof. Let $S = \{v_1, v_2, v_3, \ldots, v_s\}$ a resolving set of Q^n. We first define $\theta : V^n \to A$ where $A = \{\langle d(x,v_1), d(x,v_2), \ldots, d(x,v_s)\rangle : x \in V^n\}$ by y a $\langle d(y,v_1), d(y,v_2), \ldots, d(y,v_s)\rangle$. Then,

$$|V^n| \leq |A| \Rightarrow \quad \beta_n \geq \frac{n\log 2}{\log(n+1)}.$$ W

Lemma 2.2. If u and v are vertices of Q^n, then the hamming distance from u to v denoted, $d(u,v) = \|u-v\|^2$.

Proof. Let $u, v \in V^n$. Then $d(u,v) = \sum_{j=1}^{n} |b_j - d_j| = \|u-v\|^2$. W

Lemma 2.3. Suppose that u and v are vertices of Q^n. Then $\tilde{u}\cdot\tilde{v} + 2d(u,v) = n.$

Proof. Let $u = (b_1, b_2, \ldots, b_n)$ and $v = (d_1, d_2, \ldots, d_n)$, then $u \cdot E = (b_1, b_2, \ldots, b_n)\cdot(1,1,\ldots,1) = \|u\|^2$. But, $\|E\|^2 = n.$ Thus, $\tilde{u}\cdot\tilde{v} = (2-E)\cdot(2v-E) = n - 2d(u,v).$ Therefore, $\tilde{u}\cdot\tilde{v} + 2d(u,v) = n.$ W

Theorem 2.4. (Main Theorem) The set $\{v_1, v_2, \ldots, v_m\}$ of vertices of Q^n resolve V^n if and only if the solution **z** of the linear system $z\cdot\tilde{v}_j = 0,$ where $j = 1,2,\ldots,m,$ that lies in D^n (difference vector of Q^n)is the trivial solution 0.

Proof. $(\Rightarrow)$ Suppose $\{v_1, v_2, \ldots, v_m\}$ resolves V^n.

To show: 0 is the only solution of $z\cdot\tilde{v}_j = 0,$ where $j = 1,2,\ldots,m,$ in D^n.

By *Lemma 2.3*, we have $\tilde{x}\cdot\tilde{v}_j + 2d(x,v_j) = n = \tilde{y}\cdot\tilde{v}_j + 2d(y,v_j)$

$\Leftrightarrow \quad (x-y)\cdot\tilde{v}_j = (d(y,v_j) - d(x,v_j))$.

Let $w = (w_1, w_2, w_3, \ldots, w_n)$ be a solution of $z\cdot\tilde{v}_j = 0,$. Then $w\cdot\tilde{v}_j = 0 \Rightarrow \quad 2(x-y)\cdot\tilde{v}_j = 0$ for some $x, y \in V^n \Rightarrow \quad w = 2(x-y) = 2(0) = 0.$

$(\Leftarrow)$ Suppose that 0 is a unique solution of $z\cdot\tilde{v}_j = 0,$ where $j = 1,2,\ldots,m,$.

To show: $\{v_1, v_2, \ldots, v_m\}$ resolves V^n.

Suppose $d(x,v_j) - d(y,v_j) = 0$ for all $j = 1,2,\ldots,m,$. If $z\cdot\tilde{v}_j = 0,$, then $2(x-y)\cdot\tilde{v}_j = 0.$ In order to see this, By Lemma 3.3, $\tilde{x}\cdot\tilde{v}_j + 2d(x,\tilde{v}_j) = n$ and $y\cdot\tilde{v}_j + 2d(y,\tilde{v}_j) = n$, this implies that $(\tilde{x}-\tilde{y})\cdot\tilde{v}_j + 2\left[d(x,\tilde{v}_j) - d(y,\tilde{v}_j)\right] = 0 \Rightarrow \quad (x-y) = 0$, since 0 is unique solution, this implies that $x = y$. This implies that $\{v_1, v_2, \ldots, v_m\}$ is a resolving set of V^n. W

Remark 2.5. An equation in two variables may have a unique solution in D^n. To see this, consider the equation $2x_1 + x_2 = 0$. If $2x_1 + x_2 = 0$, then $x_2 = -2x_1$, i.e. x_2 is even since $x_2 \in \{1,0,-1\}$, $x_2 = 0$. Hence, $x_1 = 0$, so $x_1 = x_2 = 0$ is a unique solution of $2x_1 + x_2 = 0$ in D^n. W

Theorem 2.6. Let $\{\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_m\} \subseteq V^n$. Consider $\{v_1, v_2, \ldots, v_m\}$. Let W be the subspace of $\mathbb{R}^n$ spanned by $\{v_1, v_2, \ldots, v_m\}$ and $W^{\perp}$ be the orthogonal complement of W in $\mathbb{R}^n$. $\{v_1, v_2, \ldots, v_m\}$ resolves V^n if and only if no non-zero difference vector in D^n. lies in $W^{\perp}$.

Proof. $(\Rightarrow)$ Suppose $\{v_1, v_2, \ldots, v_m\}$ resolves V^n, let u be a non-zero difference vector that lie in $W^{\perp}$.Then, $u\cdot\tilde{v}_1 = 0 \ldots u\cdot\tilde{v}_m = 0$. By Theorem 2.4, $u \notin D^n$. This is a contradiction.

$(\Leftarrow)$ Suppose that there is no non-zero vector that lies in $W^{\perp}$.

To show: $\{v_1, v_2, \ldots, v_m\}$ resolves V^n.

Consider the linear system

$$\begin{aligned} u\cdot\tilde{v}_1 &= 0 \\ u\cdot\tilde{v}_2 &= 0 \\ &\vdots \\ u\cdot\tilde{v}_m &= 0 \end{aligned} \qquad (1)$$

Claim: If u is orthogonal to every element of $\{v_1, v_2, \ldots, v_m\}$, then u is orthogonal to W. Let $u \in V$ with $u\cdot\tilde{v}_i = 0$ for $i = 1,2,\ldots,m$ and let $w \in W$. If $w \in W$, then $w = a_1\tilde{v}_1 + a_2\tilde{v}_2 + \ldots + a_m\tilde{v}_m$ for some $a_1, a_2, \ldots, a_m \in \mathbb{R}$. Hence, $u\cdot w = u\cdot(a_1\tilde{v}_1 + a_2\tilde{v}_2 + \ldots + a_m\tilde{v}_m) = 0$. This shows the claim.

Let x_0 be a solution of (1), then x_0 is orthogonal to $\{v_1, v_2, \ldots, v_m\}$. Hence by the claim, x_0 is orthogonal to W, i.e, $x_0 \in W^{\perp}$. Therefore, $x_0 = \vec{0}$. Accordingly, by *Theorem 2.4*, $\{v_1, v_2, \ldots, v_m\}$ resolves V^n. ∎

Theorem 2.7. The set $\{v_1, v_2, \ldots, v_m\}$ spans $\mathbb{R}^n$.

Proof. Suppose $\{v_1, v_2, \ldots, v_m\}$ spans $\mathbb{R}^n$.

To show: $W = \mathbb{R}^n$.

$(\subseteq)$ Clearly $W \subseteq V = \mathbb{R}^n$.

$(\supseteq)$ Let $x \in \mathbb{R}^n$. Since $\{v_1, v_2, \ldots, v_m\}$ spans $\mathbb{R}^n$ $x = a_1\tilde{v}_1 + a_2\tilde{v}_2 + \ldots + a_m\tilde{v}_m$ for some $a_1, a_2, \ldots, a_m \in \mathbb{R}$. Hence, $x \in W$ (since $\{v_1, v_2, \ldots, v_m\}$ spans W). This shows the inclusion. ∎

Lemma 2.8. If $\{v_1, v_2, \ldots, v_m\}$ spans $\mathbb{R}^n$ then $\{v_1, v_2, \ldots, v_m\}$ resolves V^n.

Proof. Let $\{v_1, v_2, \ldots, v_m\}$ spans $\mathbb{R}^n$ Then by Theorem 2.7, $W = \mathbb{R}^n$. Hence, by fact (2) $\mathbb{R}^n \cap W^{\perp} = \{0\}$. Since $W^{\perp} \subseteq \mathbb{R}^n$, $W^{\perp} = \{\vec{0}\}$, i.e., there is no non-zero difference vector that lie in $W^{\perp}$ Therefore by Theorem 2.4, $\{v_1, v_2, \ldots, v_m\}$ resolves V^n.

∎

Lemma 2.9. $\{-E, 2e_1 - E, 2e_2 - E, 2e_4 - E\}$ spans $\mathbb{R}^5$, i.e. {(–1, –1, –1, –1, –1), (1, –1, –1, –1, –1), (–1, 1, –1, –1, –1), (–1, –1, 1, –1, –1), (–1, –1, –1, 1, –1)} spans $\mathbb{R}^5$,

Proof. Let $v \in \mathbb{R}^5$. Then $v = (a, b, c, d, e)$ where $a, b, c, d, e \in \mathbb{R}$. . We want to find real numbers x_1, x_2, x_3, x_4, x_5 s.t. $v = x_1(-E) + x_2(2e_1 - E) + x_3(2e_2 - E) + x_4(2e_3 - E)$ $+ x_5(2e_4 - E)$ (3)

Thus, we have the linear system

$$-x_1 + x_2 - x_3 - x_4 - x_5 = a$$
$$-x_1 - x_2 + x_3 - x_4 - x_5 = b$$
$$-x_1 + x_2 - x_3 + x_4 - x_5 = c$$
$$-x_1 + x_2 - x_3 - x_4 + x_5 = d$$
$$-x_1 - x_2 - x_3 - x_4 - x_5 = e$$

We use Gaussian elimination to solve to for the linear system. Hence, there exists

$$x_1 = e - 1/2(a + b + c + d) \in \mathbb{R}.$$
$$x_2 = -1/2(e - a) \in \mathbb{R}.$$
$$x_3 = -1/2(e - b) \in \mathbb{R}.$$
$$x_4 = -1/2(e - c) \in \mathbb{R}.$$
$$x_5 = -1/2(e - d) \in \mathbb{R}.$$

s.t. (3) holds. Since v is arbitrary, the remark follows. ∎

Theorem 2.10. $\beta_3 = 3$.

Proof: Consider the set $\{(0, e_3, e_2)\}$.

Claim: $\{(0, e_3, e_2)\}$ resolves V^3.

Let $v_1 = 0 = (0,0,0)$, $v_2 = e_3 = (0,0,1)$, $v_3 = e_2 = (0,1,0)$.

Thus, $\tilde{v}_1 = 2(0,0,0) - (1,1,1) = (-1,-1,-1)$

$$\tilde{v}_2 = 2(0,0,1) - (1,1,1) = (-1,-1,1)$$
$$\tilde{v}_1 = 2(0,1,0) - (1,1,1) = (-1,1,-1)$$

Suppose that $\{v_1, v_2, v_3\}$ resolves V^3. Then by Theorem 2.4, the only solution of the linear system $x \cdot \tilde{v}_j = 0$ for $j = 1,2,3,\ldots,n$ where $x = (x_1, x_2, x_3)$ in $D^3 \subseteq Q^3$ is 0.

However, if $x \cdot \tilde{v}_1 = 0$, $x \cdot \tilde{v}_2 = 0$, $x \cdot \tilde{v}_3 = 0$. That is, $x \cdot (-1,-1,-1) = 0$, $x \cdot (-1,-1,1) = 0$, $x \cdot (-1,1,-1) = 0$. Then,

$$-x_1 - x_2 - x_3 = 0$$
$$-x_1 - x_2 + x_3 = 0 \qquad (4)$$
$$-x_1 + x_2 - x_3 = 0$$

Using Gaussian elimination, we can solve (4). Thus, $x_1 = 0$, $x_2 = 0$, $x_3 = 0$. Hence, $x = (0,0,0) = \vec{0}$ is the only solution to the linear system $x \cdot \tilde{v}_j = 0$ for $j = 1,2,3,\ldots,n$ where $x = (x_1, x_2, x_3)$ in $D^3 \subseteq Q^3$ is 0. Therefore, by Theorem 2.4, the set $\{(0, e_3, e_2)\}$ resolves V^3. Hence, $\beta_3 = 3$. ∎

Theorem 2.11. The set $\{0, e_1, e_2, \ldots, e_{n-1}\}$ resolves V^n.

Proof: Consider the set $\{0, e_1, e_2, \ldots, e_{n-1}\}$. Note that $\{0, e_1, e_2, \ldots, e_{n-1}\}$ resolves V^n.

Let

$$v_1 = 0 = (0,0,0,\ldots,0,0,0)$$
$$v_2 = e_1 = (1,0,0,\ldots,0,0,0)$$
$$v_3 = e_2 = (0,1,0,\ldots,0,0,0).$$
$$\vdots$$
$$v_{n-1} = e_{n-1} = (0,0,0,\ldots,0,1,0)$$

Thus,

$$\tilde{v}_1 = 2(0) - E = -(1,1,1,\ldots,1,1,1) = -1,-1,-1,\ldots,-1,-1,-1$$
$$\tilde{v}_2 = 2(e_1) - E = 1,-1,-1,\ldots,-1,-1,-1$$
$$\tilde{v}_3 = 2(e_2) - E = -1,1,-1,\ldots,-1,-1,-1$$
$$\vdots$$
$$\tilde{v}_{n-1} = 2(e_{n-1}) - E = -(1,1,1,\ldots,1,1,1) = -1,-1,-1,\ldots,-1,-1,-1$$

Suppose that $\{v_1, v_2, v_3, \ldots, v_{n-1}\}$ resolves V^n. Then by theorem 2.4, the only solution of the linear system $x \cdot \tilde{v}_j = 0$ in D^n is $x = \vec{0}$. Thus,

$$-x_1 - x_2 - x_3 - \ldots - x_n = 0$$
$$x_1 - x_2 - x_3 - \ldots - x_n = 0$$
$$-x_1 + x_2 - x_3 - \ldots - x_n = 0$$
$$\vdots$$
$$-x_1 - x_2 - x_3 - \ldots + x_{n-1} - x_n = 0$$

Note that $D^n = \{(x_1, x_2, x_3, \ldots, x_n) : x_i \in \{-1,0,1\},\ j = 1,2,\ldots,n\}$.

Claim. If $x = (x_1, x_2, x_3, \ldots, x_n)$ is a solution to the above linear system, then $x_1 = x_2 = x_3 = \ldots = x_{n-1}$. Thus,

$x_1 = x_2 = x_3 = ... = x_{n-1}$. Accordingly, $x = (0,0,0,...,0) = \vec{0}$. This shows the claim. By theorem 2.4, $\{0, e_1, e_2, ..., e_{n-1}\}$ resolves V^n. Hence, $\beta_n \le n$. ∎

Lemma 2.12. $\{e_1, e_1 + e_3, e_2 + e_3\}$ does not resolve V^4.

Proof. Consider the set $\{e_1, e_1 + e_3, e_2 + e_3\}$ does not resolve V^4. Let $v_1 = e_1 = (1,0,0,0)$, $v_2 = e_1 + e_3 = (1,0,1,0)$ and $v_3 = e_2 + e_3 = (0,1,1,0)$. Thus, $\tilde{v}_1 = 2v_1 - E = 2(1,0,0,0) -(1,1,1,1) = (1,-1,-1,-1)$; $\tilde{v}_2 = 2v_2 - E = 2(1,0,1,0) - (1,1,1,1) = (1,-1,1,-1)$; $\tilde{v}_3 = 2v_3 - E = 2(0,1,1,0) - (1,1,1,1) = (-1,1,1,-1)$. Suppose that $\{v_1, v_2, v_3\}$ resolves V^4. Then by Theorem 2.4, the only solution of the linear system

$$x \cdot \tilde{v}_1 = 0$$
$$x \cdot \tilde{v}_2 = 0$$
$$x \cdot \tilde{v}_3 = 0$$

in D^n is $x = \vec{0}$, where $x = (x_1, x_2, x_3, x_4)$. However, if

$$x \cdot \tilde{v}_1 = 0$$
$$x \cdot \tilde{v}_2 = 0$$
$$x \cdot \tilde{v}_3 = 0$$

That is,

$$x \cdot (1,-1,-1,-1) = 0$$
$$x \cdot (1,-1,1,-1) = 0$$
$$x \cdot (-1,1,1,-1) = 0$$

Then,

$$\begin{aligned} x_1 - x_2 - x_3 - x_4 &= 0 \\ x_1 - x_2 + x_3 - x_4 &= 0 \qquad (5) \\ -x_1 + x_2 + x_3 - x_4 &= 0. \end{aligned}$$

Using Gaussian elimination, we solve (5). Thus, $x_4 = 0$, $x_3 = 0$, $x_1 = x_2$. Let $x_1 = t$ where $t \in \mathbb{R}$, then $x_2 = t, x_3 = 0, x_4 = 0$. Hence, the solution set of (5) is $\{(t,t,0,0) : t \in \mathbb{R}\}$. Note that $D^4 = \{x - y : x, y \in V^4\}$ Observe that, $(1,1,0,0) \in D^4$ and $(1,1,0,0)$ is a solution of (5). This is a contradiction. Hence, $\{e_1, e_1 + e_3, e_2 + e_3\}$ does not resolve V^4. ∎

Lemma 2.13. Let $x, y \in V^n$. Then $d(x,y) + d(x, E-y) = n$.

Proof. Let $x = (a_1, a_2, a_3, ..., a_n)$ and $y = (b_1, b_2, b_3, ..., b_n)$ where $a_i, b_i \in \{0,1\}$ for all $i = 1,2,3,...n$. Consider $E - y = \{1-b_1, 1-b_2, ..., 1-b_n\}$. Note that $1 - b_i = 0 \Leftrightarrow b_i = 1$ and $1 - b_i = 1 \Leftrightarrow b_i = 0$ for all $i = 1,2,3,...n$.

To show:

1) $a_i \ne b_i$ if and only if $a_i = 1 - b_i$, for all $i = 1,2,3,...n$.
2) $a_i = b_i$ if and only if $a_i \ne 1 - b_i$ for all $i = 1,2,3,...n$.

Claim 1.

$(\Rightarrow)$ Suppose $a_i \ne b_i$ *WLOG*, suppose $a_i = 0$, then $b_i = 1$. Hence, $1 - b_i = 0 = a_i$.

$(\Leftarrow)$ Suppose $a_i = 1 - b_i$. *WLOG*, suppose $a_i = 0$, then $1 - b_i = 0$.

Hence, $b_i = 1 \ne 0 = a_i$.

Claim 2. $(\Rightarrow)$ Suppose $a_i = b_i$. WLOG, suppose $a_i = 0$, then $b_i = 0$. Hence, $1 - b_i \ne 0 = a_i$.

$(\Leftarrow)$ Suppose $a_i \ne 1 - b_i$. WLOG, suppose $a_i = 0$, then $1 - b_i = 1$. Hence, $b_i = 0 = a_i$. This shows the claims.

Let $d(x,y) = k$. Then there exist $i_1, i_2, i_3, ..., i_k \in \{1,2,...,n\}$ such that $b_i \ne a_i$ for all $i = 1,2,...,k$, and $a_i = b_i$ for all $i \ne i_j$ for $j = 1,2,...,k$. By the claim, $a_{i_j} = 1 - b_{i_j}$ for all $j = 1,2,...,k$ and $a_i \ne 1 - b_i$ for all $i \ne i_j$, $j = 1,2,...,k$. Hence, $d(x, E-y) = n - k$. Accordingly, $d(x,y) + d(x, E-y) = k + (n-k) = n$.

∎

Lemma 2.14. Let $\{v_1, v_2, ..., v_m\}$ be a resolving set of V^n. Then $\{v_1, ..., v_{m-1}, E - v_m\}$ also resolves V^n.

Proof. Let $\{v_1, v_2, ..., v_m\}$ be a resolving set of V^n and $x, y \in V^n$ with $x \ne y$. Since $\{v_1, v_2, ..., v_m\}$ is a resolving set of V^n,

$(d(x,v_1), d(x,v_2), ..., d(x,v_m)) \ne (d(y,v_1), d(y,v_2), ..., d(y,v_m))$.

Consider the following cases:

Case 1. $d(x,v_m) \ne d(y,v_m)$ (WLOG, $d(x,v_m) < d(y,v_m)$).

If $d(x,v_m) = d(y,v_m)$, then by Lemma 3.13,

$d(x,v_m) + d(x, E - v_m) = n = d(y,v_m) + d(y, E - v_m)$

$\Rightarrow d(x, E - v_m) > d(y, E - v_m)$. Thus, $(d(x,v_1), ..., d(x,v_{m-1}), d(x, E - v_m)) \ne (d(y,v_1), ..., d(y,v_{m-1}), d(y, E - v_m))$.

Case 2. $d(x,v_m) = d(y,v_m)$, then $(d(x,v_1), ..., d(x,v_{m-1}), d(x, E - v_m)) \ne (d(y,v_1), ..., d(y,v_{m-1}), d(y, E - v_m))$.

Accordingly, $\{v_1, ..., v_{m-1}, E - v_m\}$ is also a resolving set of V^n. ∎

Lemma 2.15. If $\{v_1, v_2, ..., v_k, ..., v_m\}$ is a resolving set of V^n, then $\{v_1, v_2, ..., E - v_k, ..., v_m\}$ is a resolving set.

Proof. Let $\{v_1, v_2, ..., v_k, ..., v_m\}$ be a resolving set of V^n and $x, y \in V^n$ with $x \ne y$. Since $\{v_1, v_2, ..., v_k, ..., v_m\}$ is a resolving set of V^n

$(d(x,v_1), d(x,v_2), ..., d(x,v_k), ..., d(x,v_m))$
$\ne (d(y,v_1), d(y,v_2), ..., d(y,v_k), .., d(y,v_m))$.

Consider the following cases.

Case 1. $d(x,v_k) \ne d(y,v_k)$ (WLOG, $d(x,v_k) < d(y,v_k)$

If $d(x,v_k) \ne d(y,v_k)$, then by Lemma 3.13,

$d(x,v_k) + d(x, E - v_k) = n = d(y,v_k) + d(y, E - v_k)$

$\Rightarrow d(x, E - v_k) > d(y, E - v_k)$. Thus, $(d(x,v_1), ..., d(x,v_{k-1}), d(x, E - v_k)) \ne (d(y,v_1), ..., d(y,v_{k-1}), d(y, E - v_k))$

Case 2. $d(x,v_k) = d(y,v_k)$

$(d(x,v_1),\ldots,d(x,v_{k-1}),\ d(x,E-v_k)) \neq (d(y,v_1),\ldots,d(y,v_{k-1}),$ $d(y,E-v_k))$ Accordingly, $\{v_1,v_2,\ldots,E-v_k,\ldots,v_m\}$ is also a resolving set of V^n. Hence, $\beta_n \leq \beta_{(n+1)}$. W

Lemma 2.16 $\{e_1+e_2+e_3+e_4+e_5, e_1+e_2+e_3, e_2+e_4, e_2+e_3+e_5\}$ resolves V^5 i.e. $\{(1,1,1,1,1),(1,1,1,0,0),(0,1,0,1,0),(0,1,1,0,1)\}$ resolves V^5.

Proof. Consider the set $\{e_1+e_2+e_3+e_4+e_5, e_1+e_2+e_3, e_2+e_4, e_2+e_3+e_5\}$. Claim: The set $\{e_1+e_2+e_3+e_4+e_5, e_1+e_2+e_3, e_2+e_4, e_2+e_3+e_5\}$. resolves V^5. Let $v_1 = e_1+e_2+e_3+e_4+e_5 = (1,1,1,1,1)$, $v_2 = e_1+e_2+e_3 = (1,1,1,0,0)$, $v_3 = e_2+e_4 = (0,1,0,1,0)$, and $v_4 = e_2+e_3+e_5 = (0,1,1,0,1)$. Thus, $\tilde{v}_1 = 2v_1 - E = 2(1,1,1,1,1) - (1,1,1,1,1) = (1,1,1,1,1)$, $\tilde{v}_2 = 2v_2 - E = 2(1,1,1,0,0) - (1,1,1,1,1) = (1,1,1,-1,-1)$, $\tilde{v}_3 = 2v_3 - E = 2(0,1,0,1,0) - (1,1,1,1,1) = (-1,1,-1,1,-1)$, $\tilde{v}_4 = 2v_4 - E = 2(0,1,1,0,1) - (1,1,1,1,1) = (-1,1,1,-1,1)$. Suppose that $\{v_1,v_2,v_3,v_4\}$ resolves V^5. Then by Theorem 2.4, the only solution of the linear system

$$x\cdot\tilde{v}_1 = 0$$
$$x\cdot\tilde{v}_2 = 0$$
$$x\cdot\tilde{v}_3 = 0$$
$$x\cdot\tilde{v}_4 = 0$$

where $x = (x_1,x_2,x_3,x_4,x_5)$ in D^n is $x = \vec{0}$. However, if

$$x\cdot\tilde{v}_1 = 0$$
$$x\cdot\tilde{v}_2 = 0$$
$$x\cdot\tilde{v}_3 = 0$$
$$x\cdot\tilde{v}_4 = 0$$

that is,

$$x\cdot(1,1,1,1,1) = 0$$
$$x\cdot(1,1,1,-1,-1) = 0$$
$$x\cdot(-1,1,-1,1,-1) = 0$$
$$x\cdot(-1,1,1,-1,1) = 0.$$

Then,

$$\begin{aligned} x_1+x_2+x_3+x_4+x_5 &= 0\\ x_1+x_2+x_3-x_4-x_5 &= 0\\ -x_1+x_2-x_3+x_4-x_5 &= 0\\ -x_1+x_2+x_3-x_4+x_5 &= 0. \end{aligned} \quad (6)$$

Using Gaussian elimination, we solve (6). Thus, $x_1 = x_5$, $x_2 = x_5$, $x_3 = -2x_5$, $x_4 = -x_5$. Let $x_5 = t$ where $t \in \mathbb{R}$, then $x_1 = t$, $x_2 = t$, $x_3 = -2t$ and $x_4 = -t$. Hence, the solution set of (6) is $A = \{(t,t,-2t,-t,t) : t \in \mathbb{R}\} \subseteq W^{\perp}$. By fact 2, $W^{\perp} \cap D^5 = \{\vec{0}\}$. Therefore, by Theorem 2.4 $\{e_1+e_2+e_3+e_4+e_5,\ e_1+e_2+e_3, e_2+e_4,\ e_2+e_3+e_5\}$ resolves V^5. W

Lemma 2.17. The linear equations $\{(1,1,1,1,1),\ (1,1,1,-1,-1),\ (-1,1,-1,1,-1),\ (-1,1,1,-1,1)\}$ resolves V^5.

Proof. Consider the linear system,

$$\begin{aligned} x_1+x_2+x_3+x_4+x_5 &= 0\\ x_1+x_2+x_3-x_4-x_5 &= 0\\ -x_1+x_2-x_3+x_4-x_5 &= 0\\ -x_1+x_2+x_3-x_4+x_5 &= 0 \end{aligned}$$

By elimination, we have $2x_1 + 2x_2 + 2x_3 = 0$ and $-2x_1 + 2x_2 = 0$.

Thus, $4x_2 + 2x_3 = 0$ this implies that x_3 is even and $x_3=0$, since $x_3 \in \{-1,0,1\}$ this implies that $x_2 = 0$, $x_1 = 0$, $x_4 = 0$ and $x_5 = 0$. Accordingly, by Theorem 2.4, the remark follows. W

Theorem 2.18 $\beta_5 = 4$.

Proof. Consider the set $\{0,\ e_2+e_5,\ e_3+e_5,\ e_4+e_5\}$.

Claim 1. $\{0,\ e_2+e_5,\ e_3+e_5,\ e_4+e_5\}$ resolves V^5.

Let

$$v_1 = 0 = (0,0,0,0,0)$$
$$v_2 = e_2+e_5 = (0,1,0,0,1)$$
$$v_3 = e_3+e_5 = (0,0,1,0,1)$$
$$v_4 = e_4+e_5 = (0,0,0,1,1)$$

Thus, $\tilde{v}_1 = 2(0,0,0,0,0) - (1,1,1,1,1) = (-1,-1,-1,-1,-1)$, $\tilde{v}_2 = 2(0,1,0,0,1) - (1,1,1,1,1) = (-1,1,-1,-1,1)$, $\tilde{v}_3 = 2(0,0,1,0,1) - (1,1,1,1,1) = (-1,-1,1,-1,1)$, and $\tilde{v}_4 = 2(0,0,0,1,1) - (1,1,1,1,1) = (-1,-1,-1,1,1)$. Suppose that $\{v_1,v_2,v_3,v_4\}$ resolves V^5. Then by Theorem 2.4, the only solution of the linear system $x\cdot\tilde{v}_j = 0$ for $j = 1,2,3,\ldots,n$ where $x = (x_1,x_2,x_3,x_4,x_5)$ in D^5 is $x = \vec{0}$. However, if $\tilde{x}\cdot\tilde{v}_1 = 0$, $\tilde{x}\cdot\tilde{v}_2 = 0$, $\tilde{x}\cdot\tilde{v}_3 = 0$, $\tilde{x}\cdot\tilde{v}_4 = 0$ that is,

$$\tilde{x}\cdot(-1,-1,-1,-1,-1) = 0$$
$$\tilde{x}\cdot(-1,1,-1,-1,1) = 0$$
$$\tilde{x}\cdot(-1,-1,1,-1,1) = 0$$
$$\tilde{x}\cdot(-1,-1,-1,1,1) = 0$$

then,

$$\begin{aligned} -x_1-x_2-x_3-x_4-x_5 &= 0\\ -x_1+x_2-x_3-x_4+x_5 &= 0\\ -x_1-x_2+x_3-x_4+x_5 &= 0\\ -x_1-x_2-x_3+x_4+x_5 &= 0 \end{aligned} \quad (7)$$

Using Gaussian elimination, we solve (7). Then, $x_1+x_2+x_3+x_4+x_5 = 0$, $x_2+x_5 = 0$ this implies that $x_2 = -x_5$ and $x_3+x_5 = 0$ implies that $x_3 = -x_5$ and $-x_3+x_4 = 0$ implies that $x_3 = x_4 = -x_5$ and also implies that $x_1 = 2x_5$. Let $x_5 = t$ where $t \in \mathbb{R}$, then $x_1 = 2t$, $x_2 = -t$,

$x_3=-t$, and $x_4=-t$. Hence, the solution set of (7) is $A=\{(2t,-t,-t,-t,t): t\in\mathbb{R}\}\subseteq W^{\perp}$. By fact 2, $W^{\perp}\cap D^5=\{\vec{0}\}$ Therefore, by Theorem 2.4 $\{0, e_2+e_5, e_3+e_5, e_4+e_5\}$ resolves V^5. Accordingly, $\beta_5=4$. ∎

Theorem 2.19. $\beta_{m+n}\le\beta_m+\beta_n$.

Proof. Consider $\{v_1,\ldots,v_n\}$ resolves V^n and $\{u_1,\ldots,u_n\}$ resolves V^m, then $\{(v_1,0),\ldots,(v_n,0),(E,u_1),\ldots,(E,u_m)\}$ resolves V^{m+n}. If we take $s=\beta_n$ and $t=\beta_m$, we can obtain $\beta_{m+n}\le\beta_m+\beta_n$. To show that the vectors $\{(v_1,0),\ldots,(v_n,0),(E,u_1),\ldots,(E,u_m)\}$ resolves V^{m+n}. ∎

Theorem 2.20. For $n\ge 6$, $\beta_n\le n-1$.

Proof. By the previous Theorem 2.18, we have $\beta_5=4$. Recall Lindström's inequality, $\beta_{m+n}\le\beta_m+\beta_n$. By the Lindström's inequality, $\beta_n\le\beta_5+\beta_{n-5}$ for $\beta_n\le\beta_5+\beta_{n-5}\le 4+(n-5)$, since $\beta_m\le m$ (Lemma 2.15). Thus, $\beta_n\le n-1$.

∎

Theorem 2.21. If $n\ge 5$ then $\{e_1,e_2,\ldots,e_{n-1}\}$ resolves V^n, so $\beta_n\le n-1$.

Proof. Consider the set $\{e_1,e_2,\ldots,e_{n-1}\}$. Let

$$\tilde{v}_1=2e_1-E=(1,-1,-1,\ldots,-1)$$
$$\tilde{v}_2=2e_2-E=(-1,1,-1,\ldots,-1)$$
$$\vdots$$
$$\tilde{v}_{n-1}=2e_{n-1}-E=(-1,-1,-1,\ldots,-1,1,-1)$$

Claim. The solution of the linear system $x\cdot\tilde{v}_j=0$ in D^n is $\vec{0}$.

We have,

$$x_1-x_2-x_3-\cdots-x_n=0$$
$$-x_1+x_2-x_3-\cdots-x_n=0$$
$$\vdots$$
$$-x_1-x_2-x_3-\cdots+x_{n-1}-x_n=0$$

Note that $D^n=\{(x_1,x_2,\ldots,x_n): x_i\in\{-1,0,1\}$ for all $i=1,2,3,\ldots,n\}$.

Claim. If $\{(x_1,x_2,\ldots,x_n)$ is a solution to the above linear system, then $x_1=x_2=\ldots=x_{n-1}$ and $(n-3)x_1+x_n=0$. We have,

$$\left[\begin{array}{cccccc|c} 1 & -1 & -1 & \cdots & -1 & -1 & 0\\ -1 & 1 & -1 & \cdots & -1 & -1 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ -1 & -1 & -1 & \cdots & 1 & -1 & 0 \end{array}\right]$$

$$: \left[\begin{array}{ccccccc|c} 1 & 0 & 0 & \cdots & 0 & -1 & 0 & 0\\ 0 & 1 & 0 & \cdots & 0 & -1 & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 1 & -1 & 0 & 0\\ 0 & 0 & 0 & \cdots & -1 & 1 & 0 & 0 \end{array}\right]$$

$$x_1-x_{n-1}=0 \iff x_1=x_{n-1}$$
$$x_2-x_{n-1}=0 \iff x_2=x_{n-1}$$
$$\vdots$$
$$x_{n-2}-x_{n-1}=0 \iff x_{n-2}=x_{n-1}$$

that implies $x_1=x_2=x_3=\ldots=x_{n-1}$. Moreover, $x_1=\underbrace{x_1+x_1+x_n}_{n-2\,terms}\Rightarrow 0=\underbrace{x_1+x_1+\cdots+x_1+x_n}_{n-3\,terms}\Rightarrow 0=(n-3)x_1+x_n$

Since, $x\in D^n$, $x_1,x_n\in\{-1,0,1\}$. Hence, $0=(n-3)x_1+x_n$ that implies $(n-3)x_1=-x_n$ and $x_1=0$. Thus, $0=x_1=x_2=x_3=\ldots=x_{n-1}$. If $x_1=0$, then $0=(n-3)(0)+x_n$ that implies $x_n=0$. Accordingly, $x=(0,0,0,\ldots,0)=\vec{0}$. This shows the claim. By Theorem 2.4, $\{e_1,e_2,\ldots,e_{n-1}\}$ resolves V^n. ∎

Theorem 2.22. $\beta_8=6$.

Proof. Consider the set $\{e_1,e_2,e_3,e_4+e_8,e_5+e_8,e_6+e_8\}$.

Claim. $\{e_1,e_2,e_3,e_4+e_8,e_5+e_8,e_6+e_8\}$ resolves V^8.

Let $v_1=e_1=(1,0,0,0,0,0,0,0)$, $v_2=e_2=(0,1,0,0,0,0,0,0)$, $v_3=e_3=(0,0,1,0,0,0,0,0)$, $v_4=e_4+e_8=(0,0,0,1,0,0,0,1)$, $v_5=e_5+e_8=(0,0,0,0,1,0,0,1)$, and $v_6=e_6+e_8=(0,0,0,0,0,1,0,1)$. Thus,

$$\tilde{v}_1=2(1,0,0,0,0,0,0,0)-(1,1,1,1,1,1,1,1)=(1,-1,-1,-1,-1,-1,-1,-1),$$
$$\tilde{v}_2=2(0,1,0,0,0,0,0,0)-(1,1,1,1,1,1,1,1)=(-1,1,-1,-1,-1,-1,-1,-1),$$
$$\tilde{v}_3=2(0,0,1,0,0,0,0,0)-(1,1,1,1,1,1,1,1)=(-1,-1,1,-1,-1,-1,-1,-1),$$
$$\tilde{v}_4=2(0,0,0,1,0,0,0,1)-(1,1,1,1,1,1,1,1)=(-1,-1,-1,1,-1,-1,-1,1)$$
$$\tilde{v}_5=2(0,0,0,0,1,0,0,1)-(1,1,1,1,1,1,1,1)=(-1,-1,-1,-1,1,-1,-1,1)$$
$$\tilde{v}_6=2(0,0,0,0,0,1,0,1)-(1,1,1,1,1,1,1,1)=(-1,-1,-1,-1,-1,1,-1,1)$$

Suppose that $\{v_1,v_2,v_3,v_4,v_5,v_6\}$ resolves V^8. Then by Theorem 2.4, the only solution of the linear system $x\cdot\tilde{v}_j=0$ for $j=1,2,3,\ldots,n$ where $x=(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8)$ in D^8 is $x=\vec{0}$. However, if

$$x\cdot\tilde{v}_1=0$$
$$x\cdot\tilde{v}_2=0$$
$$x\cdot\tilde{v}_3=0$$
$$x\cdot\tilde{v}_4=0$$
$$x\cdot\tilde{v}_5=0$$
$$x\cdot\tilde{v}_6=0$$

that is,

$$x\cdot(1,-1,-1,-1,-1,-1,-1,-1)=0$$
$$x\cdot(-1,1,-1,-1,-1,-1,-1,-1)=0$$
$$x\cdot(-1,-1,1,-1,-1,-1,-1,-1)=0$$
$$x\cdot(1,-1,-1,1,-1,-1,-1,1)=0$$
$$x\cdot(-1,-1,-1,-1,1,-1,-1,1)=0$$
$$x\cdot(-1,-1,-1,-1,-1,1,-1,1)=0$$

then

$$\begin{aligned} x_1-x_2-x_3-x_4-x_5-x_6-x_7-x_8&=0\\ -x_1+x_2-x_3-x_4-x_5-x_6-x_7-x_8&=0\\ -x_1-x_2+x_3-x_4-x_5-x_6-x_7-x_8&=0\\ -x_1-x_2-x_3+x_4-x_5-x_6-x_7+x_8&=0\\ -x_1-x_2-x_3-x_4+x_5-x_6-x_7+x_8&=0\\ -x_1-x_2-x_3-x_4-x_5+x_6-x_7+x_8&=0. \end{aligned} \qquad (8)$$

Using Gaussian elimination, we solve (8). Then, $x_1-x_2-x_3-x_4-x_5-x_6-x_7-x_8=0$, $-x_2+x_6+x_8=0$, $-x_3+x_6+x_8=0$, $-x_4+x_6=0$, and $-x_5+x_6=0$. That implies $x_4=x_5=x_6$, $x_1=x_2=x_3$, $x_7=-2x_3-2x_6$, and $x_8=x_3-x_6$. Let $x_3=t$ and $x_6=s$ where $s,t\in\mathbb{R}$. Then, $x_1=t$, $x_2=t$, $x_3=t$, $x_4=s$, $x_5=s$, $x_6=s$, $x_7=-2t-2s$, and $x_8=t-s$. Thus, the solution set of (8) is $A=\{(t,t,t,s,s,s,-2t-2s,t-s):t,s\in\mathbb{R}\}$. Note that $D^8=\{(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8):x_i\in\{-1,0,1\},\ i=1,2,\ldots,8\}$. Hence, if $(t,t,t,s,s,s,-2t-2s,t-s)\in D^8$ then, $|-2t-2s|\le 1$ implies that $t=-s$ and $|t-s|\le 1$ implies that $s=0$. Note that $t=-s=-0=0$. If $t=-s=0$, then $-2t-2s=-2(-s)-2(s)=2(0)-2(0)=0$, $t-s=0-0=0$. Hence, the only solution is $\vec{0}=(0,0,0,0,0,0,0,0)$. Therefore, $\beta_8=6$. ∎

Theorem 2.23. If $n\ge 8$, $\beta_n\le n-2$.

Proof. By the previous Theorem 2.22, we have $\beta_8=6$. Recall Lindström's inequality, $\beta_{m+n}\le\beta_m+\beta_n$. By the Lindström's inequality, $\beta_n\le\beta_8+\beta_{n-8}$ then $\beta_n\le\beta_8+\beta_{n-8}\le 6+(n-8)$, since $\beta_m\le m$ (Lemma 2.15) $\beta_n\le\beta_8+\beta_{n-8}=n-2$. Therefore, $\beta_n\le n-2$. ∎

3. EXTENSIONS:

Theorem 3.24. $\beta_{10}=7$.

Proof: Consider the set $(e_1+e_2+e_3+e_4+e_5+e_6+e_7+e_8+e_9+e_{10}$, $e_1+e_3+e_5+e_6+e_7+e_8+e_9+e_{10}$, $e_1+e_2+e_4+e_6+e_7+e_8+e_9+e_{10}$, $e_1+e_2+e_3+e_5+e_7+e_8+e_9+e_{10}$, $e_1+e_2+e_3+e_4+e_6+e_8+e_9+e_{10}$, $e_1+e_2+e_3+e_4+e_5+e_7+e_9+e_{10}$, $e_1+e_2+e_3+e_4+e_5+e_6+e_8+e_{10})$.

Claim: The set $(e_1+e_2+e_3+e_4+e_5+e_6+e_7+e_8+e_9+e_{10}$, $e_1+e_3+e_5+e_6+e_7+e_8+e_9+e_{10}$, $e_1+e_2+e_4+e_6+e_7+e_8+e_9+e_{10}$, $e_1+e_2+e_3+e_5+e_7+e_8+e_9+e_{10}$, $e_1+e_2+e_3+e_4+e_6+e_8+e_9+e_{10}$, $e_1+e_2+e_3+e_4+e_5+e_7+e_9+e_{10}$, $e_1+e_2+e_3+e_4+e_5+e_6+e_8+e_{10})$ resolves V^{10}.

Let $v_1=(e_1+e_2+e_3+e_4+e_5+e_6+e_7+e_8+e_9+e_{10})=(1,1,1,1,1,1,1,1,1,1)$, $v_2=(e_1+e_3+e_5+e_6+e_7+e_8+e_9+e_{10})=(1,0,1,0,1,1,1,1,1,1)$, $v_3=(e_1+e_2+e_4+e_6+e_7+e_8+e_9+e_{10})=(1,1,0,1,0,1,1,1,1,1)$, $v_4=(e_1+e_2+e_3+e_5+e_7+e_8+e_9+e_{10})=(1,1,1,0,1,0,1,1,1,1)$, $v_5=(e_1+e_2+e_3+e_4+e_6+e_8+e_9+e_{10})=(1,1,1,1,0,1,0,1,1,1)$, $v_6=(e_1+e_2+e_3+e_4+e_5+e_7+e_9+e_{10})=(1,1,1,1,1,0,1,0,1,1)$, $v_7=(e_1+e_2+e_3+e_4+e_5+e_6+e_8+e_{10})=(1,1,1,1,1,1,0,1,0,1)$.

Thus,

$\tilde{v}_1=2(1,1,1,1,1,1,1,1,1,1)-(1,1,1,1,1,1,1,1,1,1)=(1,1,1,1,1,1,1,1,1,1)$,

$\tilde{v}_2=2(1,0,1,0,1,1,1,1,1,1)-(1,1,1,1,1,1,1,1,1,1)=(1,-1,1,-1,1,1,1,1,1,1)$,

$\tilde{v}_3=2(1,1,0,1,0,1,1,1,1,1)-(1,1,1,1,1,1,1,1,1,1)=(1,1,-1,1,-1,1,1,1,1,1)$,

$\tilde{v}_4=2(1,1,1,0,1,0,1,1,1,1)-(1,1,1,1,1,1,1,1,1,1)=(1,1,1,-1,1,-1,1,1,1,1)$,

$\tilde{v}_5=2(1,1,1,1,0,1,0,1,1,1)-(1,1,1,1,1,1,1,1,1,1)=(1,1,1,1,-1,1,-1,1,1,1)$,

$\tilde{v}_6=2(1,1,1,1,1,0,1,0,1,1)-(1,1,1,1,1,1,1,1,1,1)=(1,1,1,1,1,-1,1,-1,1,1)$, and

$\tilde{v}_7=2(1,1,1,1,1,1,0,1,0,1)-(1,1,1,1,1,1,1,1,1,1)=(1,1,1,1,1,1,-1,1,-1,1)$.

Suppose that $\{v_1,v_2,v_3,v_4,v_5,v_6,v_7\}$ resolves V^{10}. Then by Theorem 2.4, the only solution of the linear system $x\cdot\tilde{v}_j=0$ for $j=1,2,3,\ldots,n$ where $x=(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_{10})$ in D^{10} is $x=\vec{0}$. However, if

$$x\cdot\tilde{v}_1=0$$
$$x\cdot\tilde{v}_2=0$$
$$x\cdot\tilde{v}_3=0$$
$$x\cdot\tilde{v}_4=0$$
$$x\cdot\tilde{v}_5=0$$
$$x\cdot\tilde{v}_6=0$$
$$x\cdot\tilde{v}_7=0$$

that is,

$$x\cdot(1,1,1,1,1,1,1,1,1,1)=0$$
$$x\cdot(1,-1,1,-1,1,1,1,1,1,1)=0$$
$$x\cdot(1,1,-1,1,-1,1,1,1,1,1)=0$$
$$x\cdot(1,1,1,-1,1,-1,1,1,1,1)=0$$
$$x\cdot(1,1,1,1,-1,1,-1,1,1,1)=0$$
$$x\cdot(1,1,1,1,1,-1,1,-1,1,1)=0$$
$$x\cdot(1,1,1,1,1,1,-1,1,-1,1)=0$$

Then,

$$
\begin{aligned}
&x_1+x_2+x_3+x_4+x_5+x_6+x_7+x_8+x_9+x_{10}=0\\
&x_1-x_2+x_3-x_4+x_5+x_6+x_7+x_8+x_9+x_{10}=0\\
&x_1+x_2-x_3+x_4-x_5+x_6+x_7+x_8+x_9+x_{10}=0\\
&x_1+x_2+x_3-x_4+x_5-x_6+x_7+x_8+x_9+x_{10}=0 \qquad (9)\\
&x_1+x_2+x_3+x_4-x_5+x_6-x_7+x_8+x_9+x_{10}=0\\
&x_1+x_2+x_3+x_4+x_5-x_6+x_7-x_8+x_9+x_{10}=0\\
&x_1+x_2+x_3+x_4+x_5+x_6-x_7+x_8-x_9+x_{10}=0
\end{aligned}
$$

Using Gaussian elimination, we solve (9) . Then, $x_2+x_4=0$, $x_3+x_5=0$, $x_4+x_6=0$, $x_1+x_2+x_3+x_4+x_5+x_6+x_7+x_8+x_9+x_{10}=0$, $x_5+x_7=0$, $x_6+x_8=0$, and $x_7+x_9=0$.

Let $x_1=r, x_2=s$ and $x_3=t$ where $r,s,t\in$ ¡ . Then $x_1=r$, $x_2=s$, $x_3=t$, $x_4=-s$, $x_5=-t$, $x_6=s$, $x_7=t$, $x_8=-s$, $x_9=-t$, and $x_{10}=-r$. Thus, the solution set of (9) is $A=\{(r,s,t,-s,-t,s,t,-s,-t,-r): r,s,t\in$ ¡ $\}\subseteq W^{\perp}$. By fact 2, $W^{\perp}$ I $D^{10}=\{\vec{0}\}$. Therefore, by Theorem 2.4 the set $(e_1+e_2+e_3+e_4+e_5+e_6+e_7+e_8+e_9+e_{10}, e_1+e_3+e_5+e_6+e_7+e_8+e_9+e_{10}, e_1+e_2+e_4+e_6+e_7+e_8+e_9+e_{10}$, $e_1+e_2+e_3+e_5+e_7+e_8+e_9+e_{10}$, $e_1+e_2+e_3+e_4+e_6+e_8+e_9+e_{10}$, $e_1+e_2+e_3+e_4+e_5+e_7+e_9+e_{10}$, e_1+e_2+ ϵ $+e_4+e_5+e_6+e_8+)\epsilon$ resolves V^{10}. Hence, $\beta_{10}=7$. W

Theorem 3.25. In $n\geq 10$, then $\beta_n\leq n-3$.

Proof. By the previous Theorem 3.24, we have $\beta_{10}=7$.

Recall Lindström's inequality, $\beta_{m+n}\leq\beta_m+\beta$. By the Lindström's inequality, $\beta_n\leq\beta_{10}+\beta_{n-10}$ then $\beta_n\leq\beta_{10}+\beta_{n-10}\leq 7+(n-10)$, since $\beta_m\leq m$ (Lemma 3.15) $\beta_n\leq\beta_{10}+\beta_{n-10}=n-3$. Therefore, $\beta_n\leq n-3$. W

Theorem 3.26. $\beta_{12}=8$.

Proof. Consider the set $(e_1+e_2+e_3+e_4+e_5+e_6+e_7+e_8+e_9+e_{10}+e_{11}+e_{12}, e_1+e_3+e_4+e_6+e_7+e_8+e_9+e_{10}+e_{11}+e_{12}$, $e_1+e_2+e_4+e_5+e_7+e_8+e_9+e_{10}+e_{11}+e_{12}$, $e_1+e_2+e_3+e_5+e_6+e_8+e_9+e_{10}+e_{11}+e_{12}$, $e_1+e_2+e_3+e_4+e_6+e_7+e_9+e_{10}+e_{11}+e_{12}$, $e_1+e_2+e_3+e_4+e_5+e_7+e_8+e_{10}+e_{11}+e_{12}, e_1+e_2+e_3+e_4+e_5+e_6+e_8+e_9+e_{11}+e_{12}$, $e_1+e_2+e_3+e_4+e_5+e_6+e_7+e_9+e_{10}+e_{12})$.

Claim: The set $(e_1+e_2+e_3+e_4+e_5+e_6+e_7+e_8+e_9+e_{10}+e_{11}+e_{12}$, $e_1+e_3+e_4+e_6+e_7+e_8+e_9+e_{10}+e_{11}+e_{12}$, $e_1+e_2+e_4+e_5+e_7+e_8+e_9+e_{10}+e_{11}+e_{12}$, $e_1+e_2+e_3+e_5+e_6+e_8+e_9+e_{10}+e_{11}+e_{12}$, $e_1+e_2+e_3+e_4+e_6+e_7+e_9+e_{10}+e_{11}+e_{12}$, $e_1+e_2+e_3+e_4+e_5+e_7+e_8+e_{10}+e_{11}+e_{12}$, $e_1+e_2+e_3+e_4+e_5+e_6+e_8+e_9+e_{11}+e_{12}$, $e_1+e_2+e_3+e_4+e_5+e_6+e_7+e_9+e_{10}+e_{12})$ resolves V^{12} .

Let $v_1=(e_1+e_2+e_3+e_4+e_5+e_6+e_7+e_8+e_9+e_{10}+e_{11}+e_{12})=(1,1,1,1,1,1,1,1,1,1,1,1)$, $v_2=(e_1+e_3+e_4+e_6+e_7+e_8+e_9+e_{10}+e_{11}+e_{12})=(1,0,1,1,0,1,1,1,1,1,1,1)$ $v_3=(e_1+e_2+e_4+e_5+e_7+e_8+e_9+e_{10}+e_{11}+e_{12})=(1,1,0,1,1,0,1,1,1,1,1,1)$, $v_4=(e_1+e_2+e_3+e_5+e_6+e_8+e_9+e_{10}+e_{11}+e_{12})=(1,1,1,0,1,1,0,1,1,1,1,1)$ $v_5=(e_1+e_2+e_3+e_4+e_6+e_7+e_9+e_{10}+e_{11}+e_{12})=(1,1,1,1,0,1,1,0,1,1,1,1)$, $v_6=(e_1+e_2+e_3+e_4+e_5+e_7+e_8+e_{10}+e_{11}+e_{12})=(1,1,1,1,1,0,1,1,0,1,1,1)$ $v_7=(e_1+e_2+e_3+e_4+e_5+e_6+e_8+e_9+e_{11}+e_{12})=(1,1,1,1,1,1,0,1,1,0,1,1)$, $v_8=(e_1+e_2+e_3+e_4+e_5+e_6+e_7+e_9+e_{10}+e_{12})=(1,1,1,1,1,1,1,0,1,1,0,1)$

Thus,

$$
\begin{aligned}
\tilde{v}_1&=2(1,1,1,1,1,1,1,1,1,1,1,1)-(1,1,1,1,1,1,1,1,1,1,1,1)\\
&=(1,1,1,1,1,1,1,1,1,1,1,1)\\
\tilde{v}_2&=2(1,0,1,1,0,1,1,1,1,1,1,1)-(1,1,1,1,1,1,1,1,1,1,1,1)\\
&=(1,-1,1,1,-1,1,1,1,1,1,1,1)\\
\tilde{v}_3&=2(1,1,0,1,1,0,1,1,1,1,1,1)-(1,1,1,1,1,1,1,1,1,1,1,1)\\
&=(1,1,-1,1,1,-1,1,1,1,1,1,1)\\
\tilde{v}_4&=2(1,1,1,0,1,1,0,1,1,1,1,1)-(1,1,1,1,1,1,1,1,1,1,1,1)\\
&=(1,1,1,-1,1,1,-1,1,1,1,1,1)\\
\tilde{v}_5&=2(1,1,1,1,0,1,1,0,1,1,1,1)-(1,1,1,1,1,1,1,1,1,1,1,1)\\
&=(1,1,1,1,-1,1,1,-1,1,1,1,1)\\
\tilde{v}_6&=2(1,1,1,1,1,0,1,1,0,1,1,1)-(1,1,1,1,1,1,1,1,1,1,1,1)\\
&=(1,1,1,1,1,-1,1,1,-1,1,1,1)\\
\tilde{v}_7&=2(1,1,1,1,1,1,0,1,1,0,1,1)-(1,1,1,1,1,1,1,1,1,1,1,1)\\
&=(1,1,1,1,1,1,-1,1,1,-1,1,1)\\
\tilde{v}_8&=2(1,1,1,1,1,1,1,0,1,1,0,1)-(1,1,1,1,1,1,1,1,1,1,1,1)\\
&=(1,1,1,1,1,1,1,-1,1,1,-1,1)
\end{aligned}
$$

Suppose that $\{v_1,v_2,v_3,v_4,v_5,v_6,v_7,v_8\}$ resolves V^{12} . Then by Theorem 2.4, the only solution of the linear system $x\cdot\tilde{v}_j=0$ for $j=1,2,3,\ldots,n$ where $x=(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_{10},x_{11},x_{12})$ in D^{12} is $x=\vec{0}$. However, if

$$x \cdot \tilde{v}_1 = 0$$
$$x \cdot \tilde{v}_2 = 0$$
$$x \cdot \tilde{v}_3 = 0$$
$$x \cdot \tilde{v}_4 = 0$$
$$x \cdot \tilde{v}_5 = 0$$
$$x \cdot \tilde{v}_6 = 0$$
$$x \cdot \tilde{v}_7 = 0$$
$$x \cdot \tilde{v}_8 = 0$$

That is,

$$x \cdot (1,1,1,1,1,1,1,1,1,1,1,1) = 0$$
$$x \cdot (1,-1,1,1,-1,1,1,1,1,1,1,1) = 0$$
$$x \cdot (1,1,-1,1,1,-1,1,1,1,1,1,1) = 0$$
$$x \cdot (1,1,1,-1,1,1,-1,1,1,1,1,1) = 0$$
$$x \cdot (1,1,1,1,-1,1,1,-1,1,1,1,1) = 0$$
$$x \cdot (1,1,1,1,1,-1,1,1,-1,1,1,1) = 0$$
$$x \cdot (1,1,1,1,1,1,-1,1,1,-1,1,1) = 0$$
$$x \cdot (1,1,1,1,1,1,1,-1,1,1,-1,1) = 0$$

Then,

$$\left[\begin{array}{l|l}
x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11} + x_{12} & 0 \\
x_1 - x_2 + x_3 + x_4 - x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11} + x_{12} & 0 \\
x_1 + x_2 - x_3 + x_4 + x_5 - x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11} + x_{12} & 0 \\
x_1 + x_2 + x_3 - x_4 + x_5 + x_6 - x_7 + x_8 + x_9 + x_{10} + x_{11} + x_{12} & 0 \\
x_1 + x_2 + x_3 + x_4 - x_5 + x_6 + x_7 - x_8 + x_9 + x_{10} + x_{11} + x_{12} & 0 \\
x_1 + x_2 + x_3 + x_4 + x_5 - x_6 + x_7 + x_8 - x_9 + x_{10} + x_{11} + x_{12} & 0 \\
x_1 + x_2 + x_3 + x_4 + x_5 + x_6 - x_7 + x_8 + x_9 - x_{10} + x_{11} + x_{12} & 0 \\
x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 - x_8 + x_9 + x_{10} - x_{11} + x_{12} & 0
\end{array}\right] (10)$$

Using Gaussian elimination, we solve (10). Then, $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11} + x_{12} = 0$, $x_2 + x_5 = 0$, $x_3 + x_6 = 0$, $x_4 + x_7 = 0$, $x_5 + x_8 = 0$, $x_6 + x_9 = 0$, $x_7 + x_{10} = 0$ and $x_8 + x_{11} = 0$. Let $x_1 = q$, $x_2 = r$, $x_3 = s$, and $x_4 = t$ where $q,r,s,t \in \square$. Then, $x_1 = q$, $x_2 = r$, $x_3 = s$, $x_4 = t$, $x_5 = -r$, $x_6 = -s$, $x_7 = -t$, $x_8 = r$, $x_9 = s$, $x_{10} = t$ and $x_{12} = -q - s - t$. Thus, the solution of the set of (10) is $A = \{(q,r,s,t,-r,-s,-t,r,s,t,-r,-q-s-t) : q,r,s,t \in \square\} \subseteq W^{\perp}$. By fact 2, $W^{\perp} \cap D^{12} = \{\vec{0}\}$. Therefore, by Theorem 2.4 the set ($e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 + e_8 + e_9 + e_{10} + e_{11} + e_{12}$, $e_1 + e_3 + e_4 + e_6 + e_7 + e_8 + e_9 + e_{10} + e_{11} + e_{12}$, $e_1 + e_2 + e_4 + e_5 + e_7 + e_8 + e_9 + e_{10} + e_{11} + e_{12}$, $e_1 + e_2 + e_3 + e_5 + e_6 + e_8 + e_9 + e_{10} + e_{11} + e_{12}$, $e_1 + e_2 + e_3 + e_4 + e_6 + e_7 + e_9 + e_{10} + e_{11} + e_{12}$, $e_1 + e_2 + e_3 + e_4 + e_5 + e_7 + e_8 + e_{10} + e_{11} + e_{12}$, $e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_8 + e_9 + e_{11} + e_{12}$, $e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 + e_9 + e_{10} + e_{12}$) resolves V^{12}. Hence, $\beta_{12} = 8$. W

Theorem 3.27: In $n \geq 12$, then $\beta_n \leq n - 4$.

Proof: By Theorem 3.26, we have $\beta_{12} = 8$. Recall Lindström's, $\beta_{m+n} \leq \beta_m + \beta_n$. By the Lindström's inequality, $\beta_n \leq \beta_{12} + \beta_{n-12}$ then $\beta_n \leq \beta_{12} + \beta_{n-12} \leq 8 + (n-12)$, since $\beta_m \leq m$ (Lemma 3.15) $\beta_n \leq \beta_{12} + \beta_{n-12} = n - 4$. Therefore, $\beta_n \leq n - 4$. W

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