FUZZY SEPARATION AXIOMS MODULO IDEALS

R. Jehangir[1] and M. Khan[2]
[1]Department of Mathematics, Preston University H-8/4 Islamabad.
[2]Department of Mathematics, Park Road, Chak Shehzad, Islamabad.
Email: rehmanjehangir@hotmail.com; moiz@comsats.edu.pk

ABSTRACT. *In this paper, fuzzy ϑ-normal and fuzzy ϑ-regular spaces are defined, in fuzzy ideal topological spaces. Sarkar [8], investigated these notions by use of fuzzy ideals and its application like the fuzzy local function. Moreover, fuzzy homeomorphism is used over the fuzzy ideals.*

Key Words: Fuzzy ϑ-topological spaces, fuzzy local function, fuzzy-homeomorphism, fuzzy ϑ-normal and fuzzy ϑ-regular function.

1. INTRODUCTION

Zadeh [13] introduced the notion of fuzzy sets. Since then many mathematicians, have used this notion in their respective fields of study. Chang [1] introduced this notion in topology and laid down the foundation of fuzzy topological space and revealed the more appealing approach to the abstract knowledge of general topology. An alternate definition was given by Lowen [6]. Yalyac [15], used the concept of fuzzy sets and function on spaces. On the other hand, some worthwhile properties in topological ideal were obtained by delineate of Vaidyanathaswamy [12] and Kuratowski [5] in 1945 and 1966, respectively. Since 1960, Vaidyanathaswamy [11], along with many other writers investigated this notion. In 1997, Sarkar [9], presented the concept of fuzzy ideal and fuzzy ideal topological spaces by considering fuzzy sets. The subject of fuzzy ideal topological spaces became endeavoring subject for Nasef and Mahmoud [8] in 2000s. In [8], Nasef and Mahmoud defined an operator by using fuzzy ideal. Since then various fuzzy topological concepts have been studied via fuzzy ideals. This working was carried out because of the Zadeh's postulate that any logical system can be fuzzified. The idea of this paper was drawn by studying the research work of Hutton [6] and Tapi [10], who gave some worthwhile results of fuzzy normal and fuzzy regular spaces, respectively.

2.PRELIMINARIES

Throughout this paper, X represents a nonempty fuzzy set and fuzzy subset A of X, denoted by $A\leq X$, that is characterized by a membership function in the sense of Zadeh [13]. The basic fuzzy sets of X are the empty set 0_X the whole set 1_X and the class of all fuzzy sets I^X. We consider $I=[0,1]$ and I^X as the collection of fuzzy sets mapped from X to [0,1]. A subfamily of I^X satisfying the certain conditions are called fuzzy topology by Chang [1]. A fuzzy topological space, in Change's sense is denoted by (X, τ) For a fuzzy set A in a topological space, we denote cl(A), Int(A) and $1_x - A$, respectively, to denote the closure, interior and complement of A. A fuzzy point in X with support $x\in X$ and value $(0<1)$ is denoted by x_α, where x is represented by support of fuzzy point. Also, for a fuzzy point x and a fuzzy set A we shall write $x\in A$ to mean that $A(x)$ (where, $A(x)$, is the value of a fuzzy set A for some $x\in X$).

Definition 2.1. [1]. A fuzzy topology on X is a collection of fuzzy subsets of X which satisfy the following conditions: (1). 0_X, $1_X \in\tau$.

(2). If A, B$\in\tau$ then min(A(x),B(x))$\in\tau$.

(3). $A_\alpha\in\tau$ for each $\alpha\in\Delta$ then, Sup{ A_α: $\alpha\in\Delta$}$\in\tau$.

Then the pair (X, τ) is called a fuzzy topological space and its elements are called the fuzzy open sets and its compliments are called fuzzy closed. Here we will give some worthwhile functions of fuzzy sets. Let A and B be fuzzy sets in a space X. Then along with many other properties, we have the following;

(i). A=B if and only if *A(x)=B(x).*

(ii). $C=A\cup B$ if and only if $C(x)=max(A(x), B(x))$

(iii). $D=A\cap B$ if and only if $D(x)=min((A(x), B(x))$.

(iv). $E=A^c$ if and only if $E(x)=1_X - A(x)$

More generally, for a family of fuzzy sets $A=\{A_\alpha: \alpha\in\Delta\}$ the union $C=\cup_{\alpha\in\Delta} A_\alpha$ and the intersection $D=\cap_{\alpha\in\Delta} A_\alpha$ are respectively defined by,

$$C(x)=\text{Sup}\{A_\alpha:\ \alpha\in\Delta\}$$

and

$$D(x)=Inf\ \{A_\alpha:\ \alpha\in\Delta\}$$

The symbol ϕ will be used to define the empty fuzzy set and ϕ(x)= 0_X, for all $x\in X$.

Lemma 2.1.[1]. Let $p:Y\to Z$, represents function and $M\in I^Y$ and $N\in I^Z$. Then

(01) $M \geq p(p^{-1}(M))$ with equality if p is surjective,

(02). $N\leq p^{-1}(p(N))$ with equality if p is injective.

(03). $p^{-1}(N^c) = p^{-1}(N)^c$.

Suppose that, Y and Z are any sets, and $f:Y\to Z$ and $A\in I^Y$ and $D\in I^Z$. Then $f(A)$ is defined to be fuzzy set in Y, defined by $f(A) = \begin{cases} sup\ A(z);\ where\ z\in f^{-1}(y),\ if\ f^{-1}(y)\neq 0_X \\ 0; \quad otherwise\end{cases}$

for $y\in Z$, and $f^{-1}(D)$ will be a fuzzy set in Y, defined as, $f^{-1}(D)(x) = D(f(x))$, $x\in Y$.

Lemma 2.2: [2]. If (X,τ,ϑ) is a fuzzy ideal topological space, then ϑ is fuzzy codense if and only if $G < G^*$ for every fuzzy open set G in I^X..

Definition 2.3: [9]. A mapping $\vartheta: I^X \longrightarrow I$ is called fuzzy ideal on X, if and only if the following conditions are satisfied,

(01). If $\lambda\leq\mu$, then $\vartheta(\lambda)\geq\vartheta(\mu)$, for each $\lambda, \mu\in I^X$.

(02). For each $\lambda,\mu\in I^X$, $\vartheta(\lambda\wedge\mu)\geq\vartheta(\lambda)\vee\vartheta(\mu)$.

***Definition* 2.4.** [13]. Then f is called fuzzy homeomorphism if and only if both f and f^{-1} are fuzzy continuous. A property of a space is called homeomorphic invariant if and only if it is invariant under fuzzy homeomorphisms. It is well known that the simplest fuzzy ideals on X and 0_X, 1_X and I^X. The triple (X,τ,ϑ) means a fuzzy topological space with a fuzzy ideal ϑ and fuzzy topology. In (X,τ,ϑ), for a fuzzy subset $A\leq X$,

$A^*(\tau,\vartheta) = \vee\{x\in X:\ A\wedge U\in\vartheta,\ for\ every\ U\in\tau\}$

is called the fuzzy local function of A with respect to ϑ [9].

While $A^*(\tau,\vartheta)$ (A^* or $A^*(\vartheta)$) is the union of the fuzzy points such that if $U\in\tau$ and $E\in I$. Fuzzy closure operator of a fuzzy set A in (X,τ,ϑ) is defined as $Cl^*(A) = A\vee A^*$.

3. FUZZY ϑ-NORMAL SPACE

In this section, we will introduce the notion of fuzzy ϑ-normal spaces and explore the characteristics of it.

Definition 3.1: Let (X,τ,ϑ) be fuzzy ideal topological space and A and B be any two disjoint fuzzy closed sets of X having disjoint fuzzy open sets U and V with the given condition that $A\leq U$ and $B\leq V$, respectively.

Definition 3.2. An fuzzy ideal topological space (X,τ,ϑ) is said to be fuzzy ϑ-normal ideal topological space if for every pair of disjoint closed fuzzy sets A and B of X, there exist disjoint fuzzy open sets U and V such that, $min(A, U^c)\in\vartheta$ and $min(B, V^c)\in\vartheta$.

Remark 3.3. Let (X,τ,ϑ) be fuzzy ideal topological space. If $\vartheta = 0_X$ then fuzzy ϑ-normality coincides fuzzy normality.

Proof. Since (X,τ,ϑ) is fuzzy ϑ-normal space and $\vartheta = 0_X$ then by hypothesis,

$$min(A, U^C) \in \vartheta = 0_X$$
$$\Rightarrow A \wedge U^C = 0_X$$
$$\Rightarrow A \leq U$$

Similarly, it can be shown that $B \leq V$. Which is the required condition for fuzzy normality of fuzzy ideal topological space.

Theorem 3.5. Let (X,τ,ϑ) be fuzzy ideal topological space. Then the following are equivalent.

(a). (X,τ,ϑ) is I-normal.

(b). For every fuzzy closed set F and fuzzy open set G containing F, there exists a fuzzy open set V such that $\min(F, V^C) \in \vartheta$ and $\min(cl(V), G^C) \in \vartheta$.

(c). For each pair of disjoint fuzzy closed sets A and B, there exists an fuzzy open set U such that $min(A, U) \in \vartheta$ and $\min(cl(U), B) \in \vartheta$.

Proof. $(a) \rightarrow (b)$. Let F by fuzzy closed set and G be fuzzy open set such that $F < G$. Then G^C is a fuzzy closed set such that $min(G^C, F) = 0_X$. By hypothesis, there exist disjoint fuzzy open sets U and V such that $min(G^C, U^C) \in \vartheta$ and $min(F, U^C) \in \vartheta$.

Now, $min(U, V) = 0_X$

$$\Rightarrow cl(V) < U^C$$

And, $min(G^C, cl(V)) < \min(G^C, U^C)$

So, $\Rightarrow min(G^C, cl(V)) < \min(G^C, U^C) \in \vartheta$

Therefore, $min(G^C, cl(V)) \in \vartheta$.

$(b) \rightarrow (c)$. Let A and B be disjoint fuzzy closed subsets of X. Then there exists an fuzzy open set U such that $min(A, U^C) \in \vartheta$ and $min(cl(U), B^C) \in \vartheta$ which implies that $min(A,U^C) \in \vartheta$ and $min(cl(U), B) \in \vartheta$.

$(c) \rightarrow (a)$. Let A and B be disjoint fuzzy closed subsets in X. Then there exists a fuzzy open set U such that $min(A, U^C) \in \vartheta$ and $min(clU), B,) \in \vartheta$ implies that $min(B, (cl(U)^C)^C) \in \vartheta$. If $V = (cl(U))^C$, then V is a fuzzy open set.

***Corollary* 3.6.** Let (X,τ,ϑ) *be a* fuzzy ideal topological space *where* ϑ *be* fuzzy codense. Then the following are equivalent.

(a) (X,τ,ϑ) is fuzzy ϑ–normal spaces.

(b) For every fuzzy closed set **F** and fuzzy open set **G** containing F, there exists a fuzzy open set **V** such than $\min(F, V^c) \in \vartheta$ and $\min(V^*, G^c) \in \vartheta$.

(c) For each pair of disjoint fuzzy closed sets A and B, there exists a fuzzy open set U such that $\min(A, U^c) \in \vartheta$ and $\min(U^*, B) \in \vartheta$.

If ϑ is a fuzzy ideal of fuzzy subsets of **X** and **Y** is a subset of **X**, then $\vartheta_Y = \{Y \cap I : I \in \vartheta\} = \{I \in \vartheta : I \subset Y\}$ is a fuzzy soft ideal of subsets of Y. The following theorem shows that fuzzy ϑ– normality is fuzzy soft closed hereditary. Since every fuzzy soft topological space (X,τ) is the fuzzy soft ideal topological space (X,τ,ϑ) where $\vartheta = 0_X$, it follows that the condition soft closed on the soft subset cannot be dropped.

***Theorem* 3.7.** If (X,τ,ϑ) is a fuzzy soft ϑ –normal space on a fuzzy soft ideal topological space and $Y \subset X$ is fuzzy soft closed, then (Y,τ_Y,ϑ_Y) is fuzzy soft ϑ_Y– normal space.

Proof. Let A and B be disjoint τ_Y fuzzy soft closed subsets of . Since **Y** is fuzzy soft closed, **A** and **B** are disjoint fuzzy soft closed subsets of **X**. By hypothesis, there exist disjoint fuzzy soft open sets U and V such that $min(A, U^C) \in \vartheta$

$$min(B,V^C) \in \vartheta . min(A,U^C) = I \in \vartheta \text{ and } min(B,V^C) = j \in \vartheta.$$

Then, $max(U,I)$ and $B < max(V,j)$.

Since, $A < Y$,

$$A < min(Y, max(U,I))$$
$$\text{and} \quad A < max(min(Y,U), min(Y,I))$$

Therefore, $min(A, (min(Y,U))^C)(Y,I) \in \vartheta_Y$

Similarly,

$\min(B, (min(Y,V))^C) < min(Y,j) \in \vartheta_Y$

If $U_1 = \min(Y,V)$, then U_1 and V_1 are disjoint τ_Y fuzzy soft open sets such that $\min(A,U_1) \in \vartheta_Y$ and in $(B,V_1) \in \vartheta_Y$. Hence, (Y,τ_Y,ϑ_Y) is fuzzy soft ϑ_Y–normal space. If (X,τ,ϑ) is a fuzzy soft ideal topological space and $f : (X,\tau,\vartheta) \rightarrow (Y,\sigma)$ is a function, then $f(\vartheta)$ is a fuzzy soft ideal on (Y,σ). In the following theorem, we will show that fuzzy soft ϑ– normality is a fuzzy soft topological property.

Theorem 3.8. If (X,τ,ϑ) is fuzzy soft ϑ –normal space and

$$f: (X,\tau,\vartheta) \rightarrow (Y,\sigma, f(\vartheta))$$

is a fuzzy soft homeomorphism, then $(Y,\sigma, f(\vartheta))$ is a fuzzy soft ideal on Y. The following theorem shows that fuzzy soft ϑ-normality is a soft topological property.

Proof. Let **A** and B be disjoint fuzzy soft σ-closed subsets of Y. Since f is fuzzy continuous, $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint fuzzy soft closed subsets of X. Since (X,τ,ϑ) is fuzzy soft ϑ –normal space, there exist disjoint fuzzy soft open sets U and V in X such that

$$min(f^{-1}(A), U^C) \in \vartheta,$$
$$min(f^{-1}(B), V) \in \vartheta$$
$$\Rightarrow f(min(f^{-1}(A), (U)^C) \in f(\vartheta)$$
$$\Rightarrow min(A, (f(U))^C) \in f(\vartheta).$$

4.0 FUZZY ϑ-REGULAR SPACE

In this section, we will introduce the notion of fuzzy ϑ-regular spaces and explore the characteristics of it.

Definition 4.1. A fuzzy ideal topological space (X,τ,ϑ) is said to be ϑ-regular for any fuzzy closed set F and fuzzy point x such that $x \notin F$, then there exists disjoint fuzzy open sets U and V such that $x \in U$ and $min(F, V^c) \in \vartheta$.

Also, this is to be noted that ϑ-normality and ϑ-regularity are independent concepts and for T_1 space, ϑ-regularity implies ϑ-normality.

Theorem 4.2. Let (X,τ,ϑ) be fuzzy ideal topological space. Then the following are equivalent,

1. X is fuzzy ϑ-regular.
2. for each $x \in X$ and fuzzy open set U containing x, there is a fuzzy open set V containing x such that $\min(Cl(V), U^c) \in \vartheta$.
3. *for* each $x \in X$ and fuzzy closed set A not containing x, there is a fuzzy open set V containing x such that $\min(cl(V), A) \in \vartheta$.

Proof. $(a) \Rightarrow (b)$ let $x \in X$ and U be a fuzzy open set containing x. Then, there exist disjoint fuzzy open sets V and W such that $x \in V$ and $\min(U^c, W^c) \in \vartheta$. If $\min(U^c, W^c) = I \in \vartheta$, then $U^c < max(W, I)$. Now $\min(V,W) = 0_X$. implies that $V < W^c$ and so $cl(V) < W^c$.

Now $\min(Cl(V), U^c) < min(W^c, max(W, I)) = min(W^c, I) < I \in \vartheta$.

$(b) \Rightarrow (c)$. Let A be fuzzy closed in X such that $x \notin A$. Then, there exists a fuzzy open set V containing x such that $min(Cl(V), A^c) \in \vartheta$.

$(c) \Rightarrow (a)$. Let A be fuzzy closed in X such that $x \notin A$. Then, there is a fuzzy open set V containing x such that

$$min(Cl(V), A) \in \vartheta. \text{ If } min(Cl(V), A) = I \in \vartheta,$$

then, $\min(A, ((Cl(V))^c)^c) = I \in \vartheta$. V and $(Cl(V))^c$ are the required disjoint fuzzy open sets such that $x \in V$ and $\min(A, ((Cl(V))^c)^c) \in \vartheta$. Hence X is ϑ-regular.

Definition 4.3. A fuzzy ideal topological space (X,τ,ϑ) is said to be ϑ-paracompact if for every $\cup$ fuzzy open cover 0 of X, there exists a locally finite refinement $\aleph$ such that $(V/V \in \aleph)^C \in \vartheta$.

Theorem 4.4. If (X,τ,ϑ) is a fuzzy ϑ-paracompact, Hausdorf space, then (X,τ,ϑ) is ϑ-normal.

Proof. Let A and B be disjoint fuzzy closed subsets of X. Since (X, τ, ϑ) is ϑ-regular, for each $x \in A$, there exist disjoint fuzzy open set U_x and V_x such

Corollary 4.5. If (X, τ, ϑ) is ϑ-compact and Hausdorf, then (X,τ,ϑ) is ϑ-normal.

Theorem 4.6. If (X, τ, ϑ) is a Lindelof, ϑ- regular space, then (X, τ, ϑ) is ϑ-normal.

Proof. Let A and B be two disjoint fuzzy closed subsets of X. Since (X, τ, ϑ) is ϑ-regular, for each $a \in A$, there is a fuzzy open set U_α such that $a \in U_\alpha$ and $min(Cl(U_\alpha), B) \in \vartheta$. Since the collection $min\{(U_\alpha, A): \alpha \in A\}$ is a cover of A by fuzzy open subsets of A and A is a Lindelof subspace of X, A = sup {Ui: $i \in N$} . Also min $(Cl(U_i), B) \in \vartheta$ for every $i \in \vartheta$. Similarly, we can find a countable collection $\{V_i : i \in N\}$ of fuzzy open sets such that $B < sup\{V_i : i \in N\}$ and $min\{Cl(V_i), A\} = I_i \in \vartheta$ for every $i \in N$. For each $n \in N$, let $G_n = \min(U_n, (\sup(Cl(Vi: i = 1, 2, 3, ..., n))))$ and $H_n = \sup\{H_n : n \in N\}$. Since G_n and H_n are fuzzy open for each $n \in N$, G and H are fuzzy open subsets of X. Clearly, $\min(G, H) = 0_X$. Now to prove that $\min(A, G^c) \in \vartheta$. Let $x \in A$. Then $x \in U_m$ for some m. Also, $\min(Cl(V_n, A), A) = I_n \in \vartheta$ for every n implies that $A < \max(In,(Cl(V_n))^c)$ for every n. Therefore, $x \in I_n$ or x $Cl(V_n)$ for every n. Hence $x \in A$ implies that $x \in G_m$, $x \in G$ and so $x \in \max(G, I)$. Hence $A < \max(G, I)$ which implies that $\min(A, G^c) < I \in \vartheta$. Since $x \in Gm$, $x \in G$ and so $x \in \max(G, I)$. Hence $A < \max(G, I)$ which implies that $\min(A, G^c) = I \in \vartheta$.

Similarly, we can prove that $\min(B, H^c) \in \vartheta$. Hence (X, τ, ϑ) is ϑ–normal.

Conclusion: In this small stint, we have fuzzified the abstract knowledge of general topology in fuzzy form. This is a tool or a platform to provide the application form of set theory. In work, can be expanded in rest of the separation axioms and other abstract themes and phenomenon can be developed on this pattern.

REFERENCES

[1] Chang CL. Fuzzy topological spaces. J Math Anal Appl. 1968;24.

[2] Hatir E, Noiri T. On -I-open sets and decomposition of almost I-continuity. Bull Malays Math Sci Soc.(2) 2006;29(1).

[3] Hatir E, Jafari S. Fuzzy semi-I-open sets and fuzzy semi-I-continuity via idealization. Chaos, Solitons & Fractals 2007;34.

[4] Hayashi E. Topologies defined by local properties. Math Ann 1964;156.

[5] Kuratowski K. Topology, vol.I. New York: Academic Press; 1966. transl.

[6] B.Hutton, Normality in fuzzy topological spaces, J. Math. Anal. Appl. 50 (1975).

[7] Lowen R. Fuzzy topological spaces and fuzzy compactness. J Math Anal Appl 1976.

[8] Nasef AA, Mahmoud RA. Some topological applications via fuzzy ideals. Chaos, Solitons & Fractals 2002.

[9] M.E.EL-Shafei, I.M.Hanafy, A.I.Aggour,RS-Compactness in L-fuzzy Topological Spaces, Kyungpook Math J.42(2002), 417-428.

[10] U. D. Tapi, R. Navalakhe, Generalized Regular Fuzzy Biclosure Spaces, Katmandu University Journal Of Science, Engineering & tech., 8(2), (2012).

[11] Vaidyanathaswamy R. Set topology, New York: Chelsa; 1960.

[12] Vaidyanathaswamy R. The localization theory in set topology. Proc Indian Acad Sci 1945.

[13] Yalvac T. H., "Fuzzy sets and functions on fuzzy spaces", Journal of Mathematical Analysis and Applications, 126(2)(1987).

[14] Zadeh LA. Fuzzy Sets.Inform. and Control 1965.