ON A GENERALIZATION OF SMALL SUBMODULES

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ABSTRACT.: Let *R* be an associative ring with identity and let M be a right *R*- module . Let *A* be a submodule of *M* , we say that *A* is μ- small (denoted by $A<<_\mu M$) if whenever M = A + X , $\frac{M}{X}$ is cosingular , then M = X. In this article , we give some properties of μ- small submodules. We say that A is a μ-coclosed submodule of M denoted by ($A\leq_{\mu cc}$ M) if whenever $\frac{A}{X}$ is cosingular and $\frac{A}{X} <<_\mu \frac{M}{X}$ for some submodule X of A , we have X = A. In this paper , several properties of these submodules are given. As a generalization of hollow module, a nonzero R- module M is called μ-hollow module if every proper submodule of M is μ-small submodule of M. Also , we give a characterization of μ- hollow modules and gives conditions under which the direct sum of μ- hollow modules is μ- hollow.

Keywords. μ-small submodule , μ-coclosed submodule , μ-hollow module.

INTRODUCTION.

Throughout this paper , rings are associative with unity and modules are unital right R- modules , where R denotes such a ring and M denotes such a module. A submodule A of M is called a small submodule of M if whenever A + B= M for some submodule B of M , we have M = B ; and in this case we write A <<M, See [1].

We write E(M) , Rad(M) and Z(M) for the injective envelope , the Jacobson radical and the singular submodule of M, respectively.

For a right R- module M , Ozcan [2] , defined the submodule Z*(M) as a dual of singular submodule to be the set of all elements $m\in$ M such that *mR* is a small module.

Z*(M) = { $m\in$ M : *mR*<< E(M) }. A module M is called cosingular (non cosingular) module if Z*(M) = M (Z*(M) =0). It is clear that Rad M ≤ Z*(M).

A submodule A of M is called coclosed submodule of M denoted by ($A\leq_{cc}$ M) if whenever $\frac{A}{X} << \frac{M}{X}$ for some submodule X of A , we have X = A. See [3]. A nonzero module M is called hollow module , if every proper submodule of M is small in M. See [1].

As a generalization of small submodules , we introduced the concept of μ- small submodules. A submodule A of M is called μ- small submodule of M (denoted by $A<<_\mu$ M) if whenever M = A + X , $\frac{M}{X}$ is cosingular , then M = X.

We state the main properties of cosingular modules and introduced the main properties of μ- small submodules and supplying examples and remarks for this concepts. Also, we define μ- coclosed submodules of M and μ-hollow modules as a generalizations of coclosed submodules and hollow modules respectively and give the basic properties of these concepts and prove a characterization of μ- hollow modules and give certain conditions under which the direct sum of μ- hollow modules is μ- hollow.

2. Cosingular modules and μ-small submodules

In this section , we give the basic properties of cosingular modules , also we added some results about cosingular modules which are needed later. we introduced μ- small submodules as a generalization of small submodules which illustrated by examples and remarks and give the properties of μ- small submodules.

Lemma 2.1: [2] Let M be an R- module. Then

(1) If f : M $\to$ M' is a homomorphism of R- modules M , M'. Then f (Z*(M)) ≤ Z*(M').

(2) Let A be a submodule of M. Then Z*(A) = A $\cap$ Z*(M).

(3) Let M_i ($i\in I$) be any collection of R- modules and let $M = \bigoplus_{i\in I} M_i$. Then $Z^*(M) = \bigoplus_{i\in I} Z^*(M_i)$.

Lemma 2.2.[2] For any ring R , the class of cosingular R-modules is closed under submodules , homomorphic images and direct sums but not (in general) under essential extensions or extensions.

Corollary 2.3 [2] **.** Let R be a right cosingular ring. Then any (right) R- module is cosingular.

Corollary 2.4. Every Z- module is cosingular.

We need to prove the followings.

Proposition 2.5. Let A ≤ B ≤ M such that $\frac{M}{A}$ is cosingular , then $\frac{M}{B}$ is cosingular.

Proof. Let $f : \frac{M}{A} \to \frac{M}{B}$ be defined by , $f(m+A) = m+B$, $\forall\, m\in M$. It is clear that that f is epimorphisim. Since $\frac{M}{A}$ is cosingular , $\frac{M}{B}$ is cosingular , by lemma (2.1)

Corollary 2.6. Let A and B be submodules of an R-module M. If $\frac{M}{A}$ is cosingular , then $\frac{M}{A+B}$ is cosingular.

Proof. Clear from previous proposition.

Proposition 2.7. Let M be an R- module and let A be a submodule of M , if M is cosingular module , then $\frac{M}{A}$ is cosingular.

Proof. Let M be a cosingular module , let $\pi : M \to \frac{M}{A}$ be the natural epimorphisim , π (Z*(M)) ≤ Z*($\frac{M}{A}$) , by lemma (2.1) , hence π (M) ≤ Z*($\frac{M}{A}$) , $\frac{M}{A}$ ≤ Z*($\frac{M}{A}$). But we know that Z*($\frac{M}{A}$) is a submodule of $\frac{M}{A}$, therefore $\frac{M}{A}$ = Z*($\frac{M}{A}$). Thus $\frac{M}{A}$ is cosingular.

Proposition 2.8. Let f : M→M' be a homomorphisim and let A be a submodule of M such that $\frac{M}{A}$ is cosingular , then $\frac{f(M)}{f(A)}$ is cosingular.

Proof. From the first and third isomorphisim theorems ,

$$\frac{f(M)}{f(A)} \cong \frac{\frac{M}{Kerf}}{\frac{A}{Kerf}} \cong \frac{M}{A}$$ which is cosingular. Hence $\frac{f(M)}{f(A)}$ is cosingular.

Definition 2.9. Let M be an R- module and let A be a submodule of M , we say that A is μ- small submodule of M (denoted by A<<$_\mu$ M), if whenever M = A + X , $\frac{M}{X}$ is cosingular , then M = X.

Examples and Remarks2.10.

(1) It is clear that if A is small submodule of M , then A is μ-small submodule of M. Thus 0<<$_\mu$ M. Also, in Z_4 as Z – module { $\bar{0}, \bar{2}$ } <<$_\mu$ Z_4.

(2) The converse of (1) is not true in general. For example, Consider Z_6 as Z_6- module. Note that Z*(Z_6) = {$x \in Z_6$: $x.Z_6$<< E(Z_6)} ={$x \in Z_6$: $x.Z_6$<< Z_6} = 0 , because Z_6 is injective Z- module and the only small submodule of Z_6 is 0, then every submodule of Z_6 is noncosingular. Hence, $\frac{Z_6}{\{\bar{0},\bar{3}\}} \cong \{\bar{0},\bar{2},\bar{4}\}$ and $\frac{Z_6}{\{\bar{0},\bar{2},\bar{4}\}} \cong \{\bar{0},\bar{3}\}$ are noncosingular. Thus $\{\bar{0},\bar{3}\}$and $\{\bar{0},\bar{2},\bar{4}\}$ μ-small submodules of Z_6 but not small in Z_6.

(3) 2Z is not μ- small submodule of Z.

In the following proposition we consider condition under which μ-smallness is smallness.

Proposition 2.11. Let M be a cosingular R- module and let A be a submodule of M , then A<<$_\mu$ M if and only if A<<M.

Proof. (⇐) clear.

(⇒) Let A<<$_\mu$ M , and let U≤ M such that M = A + U , since M is cosingular module , then $\frac{M}{U}$ is cosingular , (by proposition (2.7)). But A<<$_\mu$ M , therefore M = U. Thus A<<M.

Corollary 2.12. Let M be a small R- module and let A be a submodule of M , then A<<$_\mu$ M if and only if A<<M.

Proof. (⇐) clear.

(⇒) Since M is a small module , then by [2] M is cosingular module. Thus A<<M.

Corollary 2.13. Let M be an R- module and let A be a submodule of M . If M does not contain any maximal submodule , then A<<$_\mu$ M if and only if A<<M.

Proof. Let A<<$_\mu$ M , since M does not contain any maximal submodule , then M = Rad M ≤ Z*(M) , hence M is cosingular which implies that A<<M. The converse is clear.

Now , we give some properties of the μ-small submodules.

Proposition 2.14. Let M be an R- module

(1) Let A ≤ B ≤ M. Then B<<$_\mu$ M if and only if A<<$_\mu$ M and $\frac{B}{A}$ <<$_\mu$ $\frac{M}{A}$.

(2) Let A , B be submodules of M , then A+B<<$_\mu$ M if and only if A<<$_\mu$ M and B<<$_\mu$ M. More general if A_1 , A_2 , ….., A_n are submodules M with A_i<<$_\mu$ M , ∀ $i=1,\dots,n$, then $\sum_{i=1}^{n} Ai$ <<$_\mu$ M.

(3) Let A , B be submodules of M with A ≤ B , if A<<$_\mu$ B , then A<<$_\mu$ M.

(4) Let f : M→M' be a homomorphisim such that A<<$_\mu$ M , then f(A)<<$_\mu$ M'.

(5) Let M = $M_1 \oplus M_2$ be an R- module and let $A_1 \leq M_1$ and $A_2 \leq M_2$, then $A_1 \oplus A_2$<<$_\mu$ $M_1 \oplus M_2$ if and only if A_1<<$_\mu$ M_1 and A_2<<$_\mu$ M_2.

Proof. (1) (⇒) Suppose that B<<$_\mu$ M and let U be a submodule of M such that M = A +U , $\frac{M}{U}$ is cosingular , since A ≤ B , then M = B + U , but B<<$_\mu$ M , therefore M = U. Thus A<<$_\mu$ M. Now assume that $\frac{M}{A} = \frac{B}{A} + \frac{L}{A}$, for some submodule L of M and $\frac{\frac{M}{A}}{\frac{L}{A}} \cong \frac{M}{L}$ is cosingular , by third isomorphisim theorem. Then M = B + L , but B<<$_\mu$ M , hence M = L , thus $\frac{B}{A}$ <<$_\mu$ $\frac{M}{A}$.

(⇐) Suppose that A<<$_\mu$ M and $\frac{B}{A}$ <<$_\mu$ $\frac{M}{A}$. To prove that B<<$_\mu$ M , Let M = B + U , $\frac{M}{U}$ is cosingular , hence

$$\frac{M}{A} = \frac{B}{A} + \frac{U+A}{A}, \quad \frac{\frac{M}{A}}{\frac{U+A}{A}} \cong \frac{M}{U+A}$$ is cosingular , by corollary (2.6). But $\frac{B}{A}$ <<$_\mu$ $\frac{M}{A}$, then $\frac{M}{A}$ =

$\frac{U+A}{A}$ which implies that M = U + A , since $A \ll_\mu M$, $\frac{M}{U}$ is cosingular , then M = U. It follows that $A \ll_\mu M$.

(2) ($\Rightarrow$) Suppose that $A+B \ll_\mu M$ and let M = A + U , $\frac{M}{U}$ is cosingular , then M = A+B +U, but $A+B \ll_\mu M$, therefore M = U , then $A \ll_\mu M$. Similarly , $B \ll_\mu M$.

($\Leftarrow$) Assume that $A \ll_\mu M$ and $B \ll_\mu M$, to prove that $A+B \ll_\mu M$, let M = A + B + L, $\frac{M}{L}$ is cosingular for some submodule L of M , then $\frac{M}{B+L}$ is cosingular , since $A \ll_\mu M$, then M = B + L , but $B \ll_\mu M$, therefore M = L , which means that $A+B \ll_\mu M$. By induction one can easily prove that it is true for any finite number of submodules.

(3) Suppose that $A \ll_\mu B$ and let M = A + U , $\frac{M}{U}$ is cosingular. Since $B = B \cap M = B \cap (A + U) = A + (B \cap U)$, (by modular law). Now $\frac{B}{B \cap U} \cong \frac{B+U}{U} = \frac{M}{U}$ which is cosingular , hence $\frac{B}{B \cap U}$ is cosingular. But $A \ll_\mu B$, therefore $B = B \cap U$, that is $A \le B \le U$, then M = U. Thus $A \ll_\mu M$.

(4) Let $f : M \to M'$ be a homomorphisim and let $A \ll_\mu M$. By the first isomorphisim theorem $\frac{M}{Kerf} \cong f(M)$ and $\frac{A}{Kerf} \cong f(A)$. Since $A \ll_\mu M$, then $\frac{A}{Kerf} \ll_\mu \frac{M}{Kerf}$, by (1), hence $f(A) \ll_\mu f(M) \le M'$. Therefore $f(A) \ll_\mu M'$, by (3).

(5) ($\Rightarrow$) Suppose that $A_1 \oplus A_2 \ll_\mu M_1 \oplus M_2$, let $P : M_1 \oplus M_2 \to M_1$ be the projection map , since $A_1 \oplus A_2 \ll_\mu M_1 \oplus M_2$, then $P(A_1 \oplus A_2) \ll_\mu P(M_1 \oplus M_2)$, that is $A_1 \ll_\mu M_1$. Similarly , $A_2 \ll_\mu M_2$.

($\Leftarrow$) Suppose that $A_1 \ll_\mu M_1$ and $A_2 \ll_\mu M_2$. Consider the following injection maps J_1: $M_1 \to M_1 \oplus M_2$ and J_2: $M_2 \to M_1 \oplus M_2$, since $A_1 \ll_\mu M_1$ and $A_2 \ll_\mu M_2$, then $A_1 \oplus \{0\} \ll_\mu M_1 \oplus M_2$ and $\{0\} \oplus A_2 \ll_\mu M_1 \oplus M_2$, which implies that $A_1 \oplus A_2 \ll_\mu M_1 \oplus M_2$.

Note. Infinite sum of μ- small submodules of a module M need not be μ- small in M as the following example shows:

Consider Q as Z- module $< \frac{p}{q} > \ll_\mu Q$, p , $q \in Z$, but $\sum_{\frac{p}{q} \in Q} < \frac{p}{q} > = \cup \{ \frac{p}{q} \} = Q$ which is not μ - small in Q.

Remark. Let $f : M \to M'$ be a homomorphisim from R- module M in M' the inverse image of μ- small submodule of M' need not be μ- small in M , as the following example shows:

Consider $\pi : Z \to \frac{Z}{2Z} \cong Z_2$ the natural epimorphism , note that $0 \ll_\mu Z_2$, but $f^{-1}(\bar{0}) = 2Z$ is not μ- small in $f^{-1}(Z_2) = Z$.

Proposition 2.15. Let M be an R- module and let $A \le B$ be submodules of M , if B is a direct summand of M and $A \ll_\mu M$, then $A \ll_\mu B$.

Proof. Suppose that $A \ll_\mu M$ and let B be a direct summand of M , $M = B \oplus B'$, for some submodule B' of M, to show that $A \ll_\mu B$, let B = A + U , $\frac{B}{U}$ is cosingular , then M = B + B' = A + U +B'. Now by the second isomorphism theorem $\frac{M}{U+B'} = \frac{A+U+B'}{U+B'} \cong \frac{B}{B \cap (U+B')} = \frac{B}{U+(B \cap B')} = \frac{B}{U}$ which is cosingular , (by modular law). Since $A \ll_\mu M$, then M = U + B'. Now $B = B \cap M = B \cap (U + B') = U + (B \cap B') = U$, (by modular law). Thus $A \ll_\mu B$.

Proposition 2.16. Let M be an R- module and let A , B and C be submodules of M with $A \le B \le C \le M$, if $B \ll_\mu C$, then $A \ll_\mu M$.

Proof. Suppose that $B \ll_\mu C$ and let M = A + U , $\frac{M}{U}$ is cosingular for some $U \le M$, since $A \le B$, then M = B + U. Now, $C = C \cap M = C \cap (B + U) = B + (C \cap U)$, by modular law. Note that $\frac{C}{C \cap U} \cong \frac{C+U}{U} = \frac{M}{U}$ which is cosingular , then $\frac{C}{C \cap U}$ is cosingular , but $B \ll_\mu C$, therefore $C = C \cap U$, then $A \le C \le U$, that is M = U. Thus $A \ll_\mu M$.

NOTE. The converse of proposition (2.16) is not true in general. For example. Consider Z_{12} as Z- module , $0 \le \{\bar{0}, \bar{4}, \bar{8}\} \le \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \overline{10}\} \le Z_{12}$. It is clear that $0 \ll_\mu Z_{12}$, but $\{\bar{0}, \bar{4}, \bar{8}\}$ is not μ-small in $\{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \overline{10}\}$, since $\{\bar{0}, \bar{4}, \bar{8}\} + \{\bar{0}, \bar{6}\} = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \overline{10}\}$, $\frac{\{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \overline{10}\}}{\{\bar{0}, \bar{6}\}}$ is cosingular but $\{\bar{0}, \bar{6}\} \neq \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \overline{10}\}$

Proposition 2.17. Let A , B and C be submodules of M with $A \le B \le C \le M$. Then $\frac{C}{A} \ll_\mu \frac{M}{A}$ if and only if $\frac{C}{B} \ll_\mu \frac{M}{B}$ and $\frac{B}{A} \ll_\mu \frac{M}{A}$.

Proof. ($\Rightarrow$) Suppose that $\frac{C}{A} \ll_\mu \frac{M}{A}$ and let $\frac{M}{B} = \frac{C}{B} + \frac{U}{B}$, $\frac{M}{U}$ is cosingular, M = C + U , then $\frac{M}{A} = \frac{C}{A}$

$+\frac{U}{A}$. Since $\frac{C}{A} <<_\mu \frac{M}{A}$, then $\frac{M}{A} = \frac{U}{A}$, M = U and hence $\frac{M}{B} = \frac{U}{B}$, thus $\frac{C}{B} <<_\mu \frac{M}{B}$. Now since $\frac{B}{A} \leq \frac{C}{A} <<_\mu \frac{M}{A}$, then $\frac{B}{A} <<_\mu \frac{M}{A}$, by proposition (2.14-1).

($\Leftarrow$) Assume that $\frac{C}{B} <<_\mu \frac{M}{B}$ and $\frac{B}{A} <<_\mu \frac{M}{A}$ and let $\frac{M}{A} = \frac{C}{A} + \frac{K}{A}$, $\frac{M}{K}$ is cosingular , M = C + K, then $\frac{M}{B} = \frac{C}{B} + \frac{K+B}{B}$, $\frac{M}{K+B}$ is cosingular , by corollary (2.6). But $\frac{C}{B} <<_\mu \frac{M}{B}$, therefore $\frac{M}{B} = \frac{K+B}{B}$, M = K+B , hence $\frac{M}{A} = \frac{K}{A} + \frac{B}{A}$. But $\frac{B}{A} <<_\mu \frac{M}{A}$, therefore $\frac{M}{A} = \frac{K}{A}$, implies that M = K. Thus $\frac{C}{A} <<_\mu \frac{M}{A}$.

Lemma 2.18. Let M be a module such that M = A + B and M = (A $\cap$ B) + C for submodules A , B and C of M. Then M = (B $\cap$ C)+ A = (A $\cap$ C) + B.

Proof. See [4 , Lemma 1.2]

We end this section by the following theorem.

Theorem 2.19. Let M = A + B be a module with $\frac{M}{B}$ cosingular. Let B $\leq$ C and $\frac{C}{B} <<_\mu \frac{M}{B}$. Then $\frac{(A\cap C)}{(A\cap B)} <<_\mu \frac{M}{(A\cap B)}$.

Proof. Let $\frac{M}{(A\cap B)} = \frac{(A\cap C)}{(A\cap B)} + \frac{U}{(A\cap B)}$, $\frac{M}{U}$ is cosingular , then M = (A $\cap$ C) + U , implies that M = C + U. By Lemma (2.18) , M = (A $\cap$ U) + C , $\frac{M}{B} = \frac{(A\cap U)+B}{B} + \frac{C}{B}$. Since $\frac{M}{B}$ is cosingular , then $\frac{(A\cap U)+B}{B}$ is cosingular. But $\frac{C}{B} <<_\mu \frac{M}{B}$, therefore M = (A $\cap$ U) + B. Again by Lemma (2.18) , M = (A $\cap$ B) + U = U. Thus $\frac{(A\cap C)}{(A\cap B)} <<_\mu \frac{M}{(A\cap B)}$

3. μ - coclosed submodules and μ- hollow modules

Definition 3.1. Let M be an R- module and let A be a submodule of M , we say that A is a μ- coclosed submodule of M denoted by ($A \leq_{\mu cc} M$) if whenever $\frac{A}{X}$ is cosingular and $\frac{A}{X} <<_\mu \frac{M}{X}$ for some submodule X of A , we have X = A.

Examples and Remarks 3.2.

(1) Consider Z_{12} as Z- module , $\{\bar{0},\bar{3},\bar{6},\bar{9}\} \leq_{\mu cc} Z_{12}$, since the only submodule X of $\{\bar{0},\bar{3},\bar{6},\bar{9}\}$ such that $\frac{A}{X}$ is cosingular and $\frac{A}{X} <<_\mu \frac{Z_{12}}{X}$ is $\{\bar{0},\bar{3},\bar{6},\bar{9}\}$.

(2) Consider Z_8 as Z- module , let A = $\{\bar{0},\bar{2},\bar{4},\bar{6}\}$, X = $\{\bar{0},\bar{4}\}$, note that A is not μ- coclosed submodule of Z_8 , since $\frac{A}{X}$ is cosingular and $\frac{A}{X} = \frac{\{\bar{0},\bar{2},\bar{4},\bar{6}\}}{\{\bar{0},\bar{4}\}} \cong \{\bar{0},\bar{2}\} <<_\mu \cong \frac{Z_8}{\{\bar{0},\bar{4}\}} Z_4$ but A $\neq$ X.

(3) Let M be an R- module and let A be a coclosed submodule of M , then A is a μ- coclosed in M. To see this , let X be a submodule of A such that $\frac{A}{X} <<_\mu \frac{M}{X}$, $\frac{A}{X}$ is cosingular. It is sufficient to show that $\frac{A}{X} << \frac{M}{X}$. Let $\frac{M}{X} = \frac{A}{X} + \frac{B}{X}$, where B is a submodule of M contains X. Note that $\frac{M}{B} = \frac{A+(B+X)}{B} \cong \frac{A}{A\cap(B+X)}$ which is cosingular , by corollary (2.6), hence $\frac{M}{B}$ is cosingular , but we have $\frac{A}{X} <<_\mu \frac{M}{X}$, therefore $\frac{A}{X} << \frac{M}{X}$, then A = X. Thus A is a μ- coclosed in M.

(4) Let M be a cosingular R- module and let A be submodule of M , then A is a μ- coclosed if and only if it is coclosed in M.

(5) Every direct summand of an R- module M is μ- coclosed.

Proposition3.3. Let A be a μ- coclosed submodule of an R- module M , if X $\leq$ A $\leq$ M and $X <<_\mu M$, then $X <<_\mu A$.

Proof. Suppose that A is a μ- coclosed in M and $X <<_\mu M$, let A = X + K , $\frac{A}{K}$ is cosingular. Since A is μ- coclosed in M , it is sufficient to show that $\frac{A}{K} <<_\mu \frac{M}{K}$, let $\frac{M}{K} = \frac{A}{K} + \frac{B}{K}$, $\frac{M}{B}$ is cosingular , then M = A + B = X + K + B = X + B. But $X <<_\mu M$ and $\frac{M}{B}$ is cosingular , therefore M = B. So we get the result.

The following proposition gives the basic properties of μ-coclosed submodules.

Proposition3.4. Let M be an R- module and let A $\leq$ B $\leq$ M . Then :

(1) If B is μ- coclosed in M, then $\frac{B}{A}$ is μ- coclosed in $\frac{M}{A}$.

(2) If A is μ- coclosed in M , then A is μ- coclosed in B. The converse is true if B is μ- coclosed in M.

(3) Let C be a μ- coclosed submodule of M , then for any A ≤ B ≤ C, $\frac{B}{A} \ll_\mu \frac{M}{A}$ if and only if $\frac{B}{A} \ll_\mu \frac{C}{A}$.

(4) If $A \ll_\mu B$ and $\frac{B}{A}$ is μ- coclosed in $\frac{M}{A}$, then B is μ-coclosed in M.

Proof. (1) Assume that B is μ- coclosed in M , let $\frac{\frac{B}{A}}{\frac{X}{A}} \ll_\mu \frac{\frac{M}{A}}{\frac{X}{A}}$ and $\frac{\frac{B}{A}}{\frac{X}{A}}$ is cosingular , where A ≤ X ≤ B ≤ M. By the third isomorphisim theorem $\frac{B}{X} \cong \frac{\frac{B}{A}}{\frac{X}{A}} \ll_\mu \frac{\frac{M}{A}}{\frac{X}{A}} \cong \frac{M}{X}$ and $\frac{B}{X}$ is cosingular. Since B is μ- coclosed in M , then B = X , so $\frac{B}{A} = \frac{X}{A}$. Thus $\frac{B}{A}$ is μ- coclosed in $\frac{M}{A}$.

(2) Suppose that A is μ- coclosed in M and let X be a submodule of A such that $\frac{A}{X} \ll_\mu \frac{B}{X}$, $\frac{A}{X}$ is cosingular , then $\frac{A}{X} \ll_\mu \frac{M}{X}$, by proposition (2.14-3). But A is μ-coclosed in M , therefore A = X. Thus A is μ- coclosed in B

For the converse , assume that A is μ- coclosed in B and B is μ- coclosed in M. Let X be a submodule of A such that $\frac{A}{X} \ll_\mu \frac{M}{X}$, $\frac{A}{X}$ is cosingular and. By (1) $\frac{B}{X}$ is μ-coclosed in $\frac{M}{X}$ and by proposition (3.3) , $\frac{A}{X} \ll_\mu \frac{B}{X}$

But A is μ- coclosed in B, therefore A = X . Thus A is μ-coclosed in M.

(3) Clear.

(4) Assume that $\frac{B}{K} \ll_\mu \frac{M}{K}$, $\frac{B}{K}$ is cosingular , then

$$\frac{B}{K+A} \cong \frac{\frac{B}{K}}{\frac{K+A}{K}} \ll_\mu \frac{\frac{M}{K}}{\frac{K+A}{K}} \cong \frac{M}{K+A} , \frac{B}{K+A}$$

is cosingular , by propositions (2.5) and (2.14-1). Since $\frac{B}{A} \leq_{\mu cc} \frac{M}{A}$ by (1) , then $\frac{B}{K+A} \leq_{\mu cc} \frac{M}{K+A}$, then B = K+A. But $A \ll_\mu B$, therefore B = K. Thus B is μ-coclosed in M.

Definition3.5. A nonzero R- module M is called μ- Hollow module if every proper submodule of M is μ-small submodule of M.

Examples and Remarks3.6.

1- Z_4 as Z- module is μ- hollow.

2- Z_6 as Z- module is not μ- hollow , since $\{\bar{0},\bar{3}\}$ and $\{\bar{0},\bar{2},\bar{4}\}$ are not μ- small in Z_6.

3- Z as Z- module is not μ- hollow , since 2z is not μ-small in Z.

4- Every simple module is a μ- hollow. For example Z_2 as Z- module.

5- It is clear that every hollow module is μ- hollow. But the converse is not true in general. For example Z_6 as Z_6- module.

6- Let M be a cosingular R- module. Then M is hollow if and only if M is μ- hollow.

The following theorem gives a characterization of μ-hollow module.

Theorem3.7. Let M be an R- module. Then M is μ-hollow if and only if every proper submodule A of M such that $\frac{M}{A}$ is cosingular is small in M.

Proof. (⟹) Let A be a proper submodule of M such that $\frac{M}{A}$ is cosingular , we have to show that A<<M. Assume that there exists $B \subset M$ such that M = A+B. Since M is a μ- hollow , then $B \ll_\mu M$ and we have $\frac{M}{A}$ is cosingular , then M = A which is a contradiction. Thus A<<M.

(⟸) To show that M is a μ- hollow , let A be a proper submodule of M. Assume that A is not μ- small in M, that is there exists a proper submodule B of M such that $\frac{M}{B}$ is cosingular and M = A+B. By our assumption B<<M , then A = M , which is a contradiction.

Proposition3.8. A nonzero epimorphic image of μ-hollow is μ- hollow.

Proof. Let $f : M \to M'$ be an epimorphisim and let M be a μ- hollow module, we have to show that M' is μ- hollow , let A be a proper submodule of M' , then $f^{-1}(A)$ is a proper submodule of M , if $f^{-1}(A) = M$, then A = M' which is a contradiction. Since M is μ- hollow , then $f^{-1}(A) <<_{\mu} M$, and hence $A <<_{\mu} M'$, by proposition(2.14-4).

Corollary3.9. Let M be a μ- hollow and let A be a submodule of M. Then $\frac{M}{A}$ is μ- hollow

Proof. Let π: M $\rightarrow \frac{M}{A}$ be the natural epimorphisim. Since M is μ- hollow , then by previous proposition $\frac{M}{A}$ is μ- hollow.

The converse of previous corollary is not true as the following example shows: Consider Z as Z- module , note that $\frac{Z}{4Z} \cong Z_4$ is μ- hollow , but Z is not μ- hollow.

Proposition3.10. Let M be an R- module and let A be a nonzero μ- hollow submodule of M, then either $A <<_{\mu} M$ or A is μ- coclosed submodule of M but not both.

Proof: Suppose that A is a nonzero μ- hollow submodule of M and A is not μ- coclosed , we have to show that $A <<_{\mu} M$. Since A is not μ- coclosed , then there exists $L \subset A$ such that $\frac{A}{L} <<_{\mu} \frac{M}{L}$ and $\frac{A}{L}$ is cosingular. To prove that $A <<_{\mu} M$, let M = A+K , $\frac{M}{K}$ is cosingular , then $\frac{M}{L} = \frac{A}{L} + \frac{L+K}{L}$, $\frac{M}{L+K}$ is cosingular , by corollary (2.6). But $\frac{A}{L} <<_{\mu} \frac{M}{L}$, therefore M = L+K. Now $A = A \cap M = A \cap (L+K) = L + (A \cap K)$, by modular law. $\frac{A}{A \cap K} \cong \frac{A+K}{K} = \frac{M}{K}$, which is cosingular. But A is μ- hollow and L is proper submodule of A , therefore $L <<_{\mu} A$, hence $A = A \cap K$, $A \le K$, M = K. Thus $A <<_{\mu} M$.

If $A <<_{\mu} M$ and A is μ- coclosed , then $\frac{A}{0} <<_{\mu} \frac{M}{0}$,by proposition (2.14-1) , implies that A = 0 , which is a contradiction.

Proposition3.11. Every nonzero μ- coclosed submodule of μ- hollow is μ- hollow.

Proof. Let M be a μ- hollow module and let A be a nonzero μ- coclosed submodule of M , let L be a proper submodule of A , since M is μ- hollow , then $L <<_{\mu} M$. But A is μ- cocloed , therefore $L <<_{\mu} A$, by proposition (3.3). Thus A is μ- hollow.

Corollary3.12. Every direct summand of μ- hollow is μ- hollow.

Proposition3.13. Let M be a cosingular R- module , let $A <<_{\mu} M$ if $\frac{M}{A}$ is finitely generated , then M is finitely generated.

Proof. Since $\frac{M}{A}$ is finitely generated , then $\frac{M}{A} = R(x_1+A)+R(x_2+A)+\ldots\ldots+R(x_n+A)$, for some $x_1,x_2,\ldots,x_n \in$ M. Claim that $M = Rx_1+Rx_2+\ldots\ldots+Rx_n$, let $m \in M$, $m+A \in \frac{M}{A} = R(x_1+A)+R(x_2+A)+\ldots\ldots+R(x_n+A)$, $m+A = r_1(x_1+A)+ r_2(x_2+A)+\ldots\ldots+ r_n(x_n+A)$, $r_i \in R$, $\forall$ i = $1,\ldots,n$. Then $m+A = r_1x_1+ r_2x_2+\ldots\ldots+ r_nx_n+A$, $m - r_1x_1+ r_2x_2+\ldots\ldots+ r_nx_n \in A$, $m - r_1x_1+ r_2x_2+\ldots\ldots+ r_nx_n = a$, for some $a \in A$, $M = \langle x_1, x_2, \ldots, x_n \rangle + A$. Since M is cosingular , then $\frac{M}{\langle x_1, x_2,,,,,, x_n \rangle}$ is cosingular. But $A <<_{\mu} M$, therefore $M = Rx_1+Rx_2+\ldots\ldots+Rx_n$. Thus M is finitely generated.

Immediately , one can easily prove the following two corollaries.

Corollary3.14. Let M be a cosingular R- module and let A be a proper submodule of M, if M is μ- hollow and if $\frac{M}{A}$ is finitely generated , then M is finitely generated.

Corollary3.15. Let M be an R- module with any factor of M is a cosingular , let A be a proper submodule of M if M is μ- hollow and $\frac{M}{A}$ is finitely generated , then M is finitely generated.

Recall that an R- module M is called V- module if every module is M- injective. R is called V-ring , if the right module R_R is a V- module, see [5].

Theorem3.16: Let R be a V- ring , then every nonzero R- module is a μ- hollow module.

Proof: Let R be a V- ring and let M be an R- module , to show that M is μ- hollow , let A be any proper submodule of M such that M = A+ B , $\frac{M}{B}$ is cosingular. Since R is V- ring , then Z*(M)=0 , for any R- module M , by [5 , theorem12], hence $Z^*(\frac{M}{B}) = B$. But $Z^*(\frac{M}{B}) = \frac{M}{B}$. Thus M =B , so the we get the result.

Example3.17. $Q = \prod_{i=1}^{\infty} Fi$, where $F_i = Z_2$. Let R be the subring of Q generated by $\oplus_{i=1}^{\infty} F_i$ and 1_Q. Then R is commutative regular ring , hence it is V- ring by [6]. Thus R_R is μ-hollow module.

Remark. A direct sum of μ- hollow modules need not be μ- hollow as the following example shows. The Z- modules $Z_3 \cong \langle \bar{4} \rangle$ and $Z_2 \cong \langle \bar{6} \rangle$ are μ-hollow, but $Z_3 \oplus Z_2 \cong Z_{12}$ is not μ- hollow module.

Let M be an R- module. Recall that a submodule A of M is called a fully invariant if $g(A) \le A$, for every $g \in$ End(M) and M is called duo module if every submodule of M is fully invariant. See [7].

Now , we give conditions under which the direct sum of μ- hollow modules is a μ- hollow.

Proposition3.18. Let M_1 and M_2 be R- modules and let $M = M_1 \oplus M_2$ such that M is a duo module. Then M is μ- hollow if and only if M_1 and M_2 are μ- hollow, provided that $A \cap M_i \neq M_i$, $i= 1,2$, $\forall$ $A \subset M$.

Proof. ($\Rightarrow$) Clear by corollary (3.12).

($\Leftarrow$) Let A be a proper submodule of M. Since M is a duo module , then $A = (A \cap M_1) \oplus (A \cap M_2)$. Hence each of A

$\cap M_1$, $A \cap M_2$ is a proper submodule of M_1 and M_2 respectively. It follows that $A \cap M_1 <<_\mu M_1$ and $A \cap M_2 <<_\mu M_2$, since M_1 and M_2 are μ- hollow. Then by proposition (2.14-5) , $A <<_\mu M$. Thus M is μ- hollow.

Recall that an R- module M is called distributive if for all A , B and C ≤M , $A \cap (B+C) = (A \cap B)+(A \cap C)$. See [8].

Proposition3.19. Let M_1 and M_2 be R- modules and let $M = M_1 \oplus M_2$ such that M is a distributive module. Then M is μ- hollow if and only if M_1 and M_2 are μ- hollow, provided that $A \cap M_i \neq M_i$, $i= 1 ,2$, $\forall$ $A \subset M$.

Proof. ($\Rightarrow$) Clear by corollary (3.12).

($\Leftarrow$) Let A be a proper submodule of M. Since M is a distributive module , then $A = (A \cap M_1) \oplus (A \cap M_2)$. Hence each of $A \cap M_1$, $A \cap M_2$ is a proper submodule of M_1 and M_2 respectively. It follows that $A \cap M_1 <<_\mu M_1$ and $A \cap M_2 <<_\mu M_2$, since M_1 and M_2 are μ- hollow. Then by proposition (2.14-5) , $A <<_\mu M$. Thus M is μ- hollow.

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