THE DYNAMICS OF PREY-PREDATOR MODEL WITH PREY REFUGE AND STAGE STRUCTURES IN BOTH POPULATIONS

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***ABSTRACT*:** In this article the structures and dynamics of prey- predator model are proposed and studied. The refuge in prey with stage - structured in prey as well as predators are considered. Lotka-Voltera type of functional response has been proposed. The conditions, which guarantee the existence of equilibrium points, have been investigated. Uniqueness and boundedness of the solution of the system are proven .The local and global dynamical behaviors are discussed and analyzed. Finally, numerical simulations are carried out not only to confirm the theoretical results obtained, but also to show the effects of variation of each parameter on our proposed model.

Keywords:-Prey-predator, functional response, refuge, stage- structure, stability analysis, Lyapunove function.

INTRODUCTION:-

The predator –prey, competitive and cooperative models have obtained much attention from several researchers [Bazykin 1998, may 2001, kot 2001].In most of these interactions between homogenous populations .However, homogenous populations are rare within our restricted living environment , Mathematical study of predator-prey has been observed in new studies, the predator- prey model with age stages are also convenient .The researchers from both ecology and mathematical modeling are working in this area [1].

Most of the species during their life span go through several stages of life .Largely of the modeling approaches drop such kind of facts .Human beings of several age level are; consider into one single reproducting group .Such kinds of modeling approach are capable to produce simple equilibrium dynamics and some times fail to satisfy the oscillatory behaviors that are observed in nature .In order to obtain these types of oscillatory behaviors several modeling approaches are considered for set up stage -structure models of ecological systems .Among several approaches of stage - structure medels the simple one consists with the division of society into two groups (juvenile) and (adult) , with assumption that only adult individual have ability to produce .

Yang and Zhong [2] have proposed and studied system of two stage -structured deterministic and stochastic predator –prey systems where immature prey is predated with Beddington – DeAngel functional response, while mature prey is predated with Holling type -II functional response.

In fact, the influence of refuge (protection), stage-structure and the multiplicity of functional responses in the prey–predator ecological system are the most important topics of benefit. In recent years, stage-structure models have been studies widely by a lot of researchers [3-5].The prey –predator models with prey refuge have been investigated by Kar [6]. Ma *et al* [7] studied the effects of prey protection on a prey – predator model with a class of function responses .Chen et al . [8] studied a predator –prey model with Holling type -II response function incorporating a constant prey refuge .

Kadim. Majeed and Naji [9] considerd a food web model consisting of two predator - one stage -structured prey involving Lotka – Volttera type of response of function with prey refuge , Majeed and Ali [10] proposed and analyzed a food chain model consisting of two predator - one stage -structured prey with refuge involving two types of functional responses Lotka-Voltera and Holling type- II.

In this article, a stage - structured in both populations prey and predator model incorporating refuge with linear response function is proposed and analyzed. The considered model consists of four nonlinear ordinary differential equations to describe the interaction among juvenile prey, adult prey and juvenile predator, adult predator. This system is analyzed by using the linear stability analysis to find the conditions for which the feasible equilibrium points are stable. Global stability conditions for proposed model are described by using appropriate Lyapunove functions

The mathematical model:-

Consider the ecological model consisting of stage - structure prey in which the prey species growth logistically in the absence of predation and stage- structure predator in which the predator decay exponentially in the absence of prey species .It is assumed that the prey population divides into two compartments : immature prey population X(t) that represents the population size at time t and mature prey population; Y(t) which denotes to population size at time t . Furthermore the population size of the immature predator at time t is denoted by Z (t), while W (t) represents the population size of mature predator at time t.

Now in order to formulate the dynamics of such system the following assumptions are considered

1. The immature prey depends completely in it's feeding on the mature prey that growth logistically with intrinsic growth rate $r > 0$ and carrying capacity $k > 0$. The immature prey individuals grown up and become mature prey individuals with grown up rate $\beta_1 > 0$ .However the immature and mature prey facing death with natural death rates $d_1 > 0$ and $d_2 > 0$ respectively.

2. There is type of protection of the prey species from facing predation by the predator with refuge rate constant $m \in (0,1)$.

3. The immature predator individuals grown up and become mature predator individuals with grown up rate $\beta_2 > 0$.

4. The predator consumes the mature prey individual according to the Lotka -Voltera type of functional response with predation rate $c_1 > 0$ and contribute a portion of such food with conversion rates $0 < e_1 < 1$.

5. Finally, in the absence of food the immature and mature predator facing death with natural death rates $d_3 > 0$ and $d_4 > 0$ respectively.

Therefore the dynamics of the above proposed model can be represented by the following set of first order nonlinear differential equations.

$$\frac{dX}{dT} = r\,Y\left(1 - \frac{Y}{k}\right) - \beta_1 X - d_1 X$$

$$\frac{dY}{dT} = \beta_1 X - c_1(1-m)YW - d_2 Y \qquad (1)$$

$$\frac{dZ}{dT} = e_1 c_1 (1-m)YW - \beta_2 Z - d_3 Z$$

$$\frac{dW}{dT} = \beta_2 Z - d_4 W .$$

With initial conditions $X(0) \geq 0, Y(0) \geq 0, Z(0) \geq O$ and $W(0) \geq 0$.

Note that the above proposed model has twelve parameters in all which make the analysis difficult .So in order to simplify the system, the number of parameters is reduced by using the following dimensionless variables and parameters:

$$t = r\,T,\ r_1 = \frac{\beta_1}{r},\ r_2 = \frac{d_1}{r},\ r_3 = \frac{d_2}{r},\ r_4 = \frac{e_1 c_1 k}{r},$$
$$r_5 = \frac{d_3}{r},\ r_6 = \frac{\beta_2}{r},\ r_7 = \frac{d_4}{r},\ x = \frac{X}{k},\ y = \frac{Y}{k},$$
$$z = \frac{c_1}{r} Z,\ w = \frac{c_1}{r} W .$$

Then the non –dimensional form of system; (1) can be written as:

$$\frac{dx}{dt} = x\left[\frac{y(1-y)}{x} - (r_1 + r_2)\right] = f_1(x, y, z, w)$$
$$\frac{dy}{dt} = y\left[\frac{r_1 x}{y} - (1-m)\,w - r_3\right] = f_2(x, y, z, w) \quad (2)$$
$$\frac{dz}{dt} = z\left[\frac{r_4(1-m)yw}{z} - (r_5 + r_6)\right] = f_3(x, y, z, w)$$
$$\frac{dw}{dt} = w\left[\frac{r_6 z}{w} - r_7\right] = f_4(x, y, z, w).$$

With

$x(0) \geq 0, y(0) \geq 0, z(0) \geq 0$ *and* $w(0) \geq 0$.

It is observed that the number of parameters have been reduced from twelve in the system (1) to eight in the system (2).

Obviously the interaction functions of the system (2) are continuous and have continuous partial derivatives on the following positive four dimensional space.

$R^4_+ = \{(x, y, z, w) \in R^4 : x(0) \geq 0, y(0) \geq 0, z(0) \geq 0, w(0) \geq 0\}$.

Therefore these functions are Lipschitzian on R^4_+, and hence the solution of the system (2) exists and is unique .Further, all the solutions of system (2) with non - negative initial conditions are uniformly bounded as shown in the following theorem.

Theorem (**1**): All the solutions of system (2) which initiate in R^4_+ are uniformly bounded.

Proof: Let $(x(t), y(t), z(t), w(t))$ be any solution of the system (2) with non -negative initial condition $(x_0, y_0, z_0, w_0) \in R^4_+$.

Now define the function : $L(t) = x(t) + y(t) + z(t) + w(t)$, and then taken the time derivative of $L(t)$ along the solution of the system (2) , we get:

$$\frac{dL}{dt} = y(1-y) - r_2 x - r_3 y - r_5 z - r_7 w$$
$$-(1-r_4)(1-m)yw.$$

So, due to the fact that the conversion rate of food from mature prey population to immature predator population cannot exceeding the maximum predation rate of mature predator population, always $r_4 < 1$, *so*,

$$\frac{dL}{dt} \leq \frac{1}{4} - H\,L, \qquad \text{where } H = \min\{r_2, r_3, r_5, r_7\},$$

Then, $\frac{dL}{dt} + H\,L \leq \frac{1}{4}$.

Again by solving this differential inequality for the initial value $L(0) = L_0$, we get**:**

$$L(t) \leq \frac{1}{4H} + \left(L_0 - \frac{1}{4H}\right)e^{-Ht}. \text{ Then, } \lim_{t\to\infty} L(t) \leq \frac{1}{4H} .$$

So, $0 \leq L(t) \leq \frac{1}{4H}$, $\forall\, t > 0$,

hence all the solutions of system (2) are uniformly bounded and the proof is complet.

The existence of equilibrium points:-

In this section, the existence of all possible equilibrium points of system (2) is discussed. It is observed that, system (2) has at most three equilibrium points, which are mentioned in the following:

◉The equilibrium point $E_0 = (0, 0, 0, 0)$, which known as vanishing point is always exists.

◉The free predators' equilibrium point $E_1 = (\bar{x}, \bar{y}, 0, 0)$, where:

$\bar{y} = \frac{r_1(1-r_3) - r_2 r_3}{r_1}$, *and* $\bar{x} = \left(\frac{r_1(1-r_3) - r_2 r_3}{r_1^2}\right) r_3$, exists uniquely in $Int.R^2_+$ (Interior of R^2_+) of $xy-plane$ if in addition of biological fact $r_3 < 1$ the necessary condition holds:

$$r_2 < \frac{r_1(1-r_3)}{r_3} . \qquad (3)$$

◉Finally, the positive (coexistence) equilibrium point $E_2(x^*, y^*, z^*, w^*)$, where:

$$z^* = \frac{r_7(r_3 y^* - r_1 x^*)}{r_6(1-m)y^*} . \qquad (4a)$$

$$y^* = \frac{(r_5 + r_6)r_7}{r_4 r_6 (1-m)}. \qquad (4b)$$

$$x^* = \frac{r_7(r_5 + r_6)[r_4 r_6(1-m) - (r_5 + r_6)r_7]}{r_4^2 r_6^2 (1-m)^2 (r_1 + r_2)} \text{ , and } (4c)$$

$$w^* = w(z^*) = \frac{r_6}{r_7} z^* = \frac{r_3 y^* - r_1 x^*}{(1-m)y^*} \qquad (4d)$$

exists uniquely in the interior R_+^4 if the following conditions hold:

$$r_4 r_6(1-m) > (r_5+r_6)r_7, and \quad (4e)$$

$$r_1 x^* < r_3 y^* \quad (4f)$$

***The local stability analysis of system* (2)**

In this section, the local stability analysis of system (2)around each of the above equilibrium points is discussed through computing the Jacobian matrix $J(x, y, z, w)$of system (2) at each of them which is given by:

$$J = [a_{ij}]_{4\times 4} \quad , \quad where \quad (5)$$

$a_{11} = -(r_1+r_2), \quad a_{12} = 1 - 2y, \quad a_{13} = a_{14} = 0, \quad a_{21} = r_1,$
$a_{22} = -r_3 - (1-m); w, \quad a_{23} = 0,$
$a_{24} = -(1-m)\ y, \quad a_{31} = 0, \quad a_{32} = r_4(1-m)w,$
$a_{33} = -(r_5+r_6), a_{34} = r_4(1-m)y, \ a_{41} = 0,$
$a_{42} = 0, \ a_{43} = r_6, \ a_{44} = -r_7.$

***The local stability analysis at* E_0 :**

The Jacobian matrix of system (2) at E_0 can be written as:

$$J_0 = J(E_0) = \begin{bmatrix} -(r_1+r_2) & 1 & 0 & 0 \\ r_1 & -r_3 & 0 & 0 \\ 0 & 0 & -(r_5+r_6) & 0 \\ 0 & 0 & r_6 & -r_7 \end{bmatrix} \quad (6a)$$

Then the characteristic equation of J_0 is given by:

$$(\lambda^2 + tr(A)\lambda + \det(A))(-(r_5+r_6)-\lambda)(-r_7-\lambda) = 0$$

$so, either$

$$(-(r_5+r_6)-\lambda)(-r_7-\lambda) = 0,$$

this gives two eigenvalues of J_0 by:

$\lambda_{0z} = -(r_5+r_6) < 0$, and $\lambda_{0w} = -r_7 < 0.$

Or

$$\lambda^2 + tr(A)\lambda + \det(A) = 0,$$

where:

$A = \begin{bmatrix} -(r_1+r_2) & 1 \\ r_1 & -r_3 \end{bmatrix}$, so,

$tr(A) = \lambda_{0x} + \lambda_{0y} = -(r_1+r_2+r_3) < 0$, and

$det(A) = \lambda_{0x} . \lambda_{0y} = r_1(r_3-1) + r_2 r_3.$

Which gives the other two eigenvalues of J_0 with negative real parts provided that the following condition holds:

$$r_3 > 1. \quad (6b)$$

Then E_0 is locally asymptotically stable in the R_+^4 . However, it is; a saddle point (unstable) otherwise.

***The local stability analysis at* E_1**

The Jacobian matrix of system (2) at E_1 can be written as:

$$J_1 = J(E_1) = [b_{ij}]_{4\times 4}, \quad (7a)$$

where:

$b_{11} = -(r_1+r_2), \ b_{12} = 1 - 2\bar{y}, \ b_{13} = b_{14} = 0,$
$b_{21} = r_1, b_{22} = -r_3, \ b_{23} = 0, \ b_{24} = -(1-m)\bar{y}, b_{31} = b_{32} = 0, b_{33} = -(r_5+r_6), \ b_{34} = r_4(1-m)\bar{y}, \ b_{41} = b_{42} = 0, b_{43} = r_6, \ b_{44} = -r_7.$

Then the characteristic equation of J_1 is given by:

$$[\lambda^2 - tr(\bar{A})\lambda + \det(\bar{A})][\lambda^2 - tr(\bar{B})\lambda + \det(\bar{B})] = 0 \quad (7b)$$

where:

$\bar{A} = \begin{bmatrix} -(r_1+r_2) & 1-2\bar{y} \\ r_1 & -r_3 \end{bmatrix}, and$

$\bar{B} = \begin{bmatrix} -(r_5+r_6) & r_4(1-m)\bar{y} \\ r_6 & -r_7 \end{bmatrix}$, so

$tr(\bar{A}) = \lambda_{1x} + \lambda_{1y} = -(r_1+r_2+r_3) < 0,$

$det(\bar{A}) = \lambda_{1x} . \lambda_{1y} = r_3(r_1+r_2) - r_1(1-2\bar{y})$,and

$tr(\bar{B}) = \lambda_{1z} + \lambda_{1w} = -(r_5+r_6+r_7) < 0,$

$det(\bar{B}) = \lambda_{1z} . \lambda_{1w} = r_7(r_5+r_6) - r_4 r_6(1-m)\bar{y}$

$so, either$

$$[\lambda^2 + tr(\bar{A})\lambda + \det(\bar{A})] = 0, \quad (7c)$$

which gives the two eigenvalues of J_1 with negative real parts due to the following condition

$$\bar{y} > \frac{1}{2} \quad (7d)$$

Or

$$[\lambda^2 + tr(\bar{B})\lambda + \det(\bar{B})] = 0$$

which gives the other two eigenvalues of J_1 with negative real parts due to the following condition

$$\bar{y} < \frac{r_7(r_5+r_6)}{r_4 r_6(1-m)}. \quad (7e)$$

Hence, $E_1$is locally asymptotically stable in R_+^4 .However, it is a saddle (unstable) point otherwise.

***The local stability analysis at* E_2**

The Jacobian matrix of system (2) at E_2 can be written as:

$$J_2 = [d_{ij}]_{4\times 4} \quad , \quad (8a)$$

$where$:

$d_{11} = -(r_1+r_2) < 0 \ , \quad d_{12} = 1 - 2y^*,$
$d_{13} = d_{14} = 0, \ d_{21} = r_1 > 0,$
$d_{22} = -r_3 - (1-m)w^* < 0, \ d_{23} = 0,$
$d_{24} = -(1-m)y^*, d_{31} = 0 \ , d_{32} = r_4(1-m)w^* > 0,$
$d_{33} = -(r_5+r_6), d_{34} = r_4(1-m)y^* > 0, d_{41} = 0,$
$d_{42} = 0, \ d_{43} = r_6 > 0, d_{44} = -r_7 < 0.$

Then the characteristic equation of J_2 is given by:

$$[\lambda^4 + B_1\lambda^3 + B_2\lambda^2 + B_3\lambda + B_4] = 0, \quad (8b)$$

where:

$B_1 = -(\gamma_0+\gamma_1) > 0.$
$B_2 = \gamma_0\gamma_1 + \gamma_3 - \gamma_5.$
$B_3 = -2\gamma_0\gamma_2 - \gamma_1(\gamma_3+\gamma_5) - \gamma_6.$
$B_4 = d_{11}\gamma_6 > 0$, with,
$\gamma_0 = (d_{11}+d_{22}) < 0 \ , \gamma_1 = d_{33} + d_{44} < 0,$
$\gamma_2 = d_{33}d_{44} = \gamma_4 = d_{34}d_{43} > 0, \ \gamma_3 = d_{11}d_{22} < 0,$
$\gamma_5 = d_{12}d_{21} > 0, \ \gamma_6 = d_{24}d_{32}d_{43} < 0.$

Now by using Routh -Hawirtiz criterion equation (8 b) has roots (eigenvalues) with negative real parts if and only if $B_i > 0, \ i = 1,3,4 \ and$

$$\Delta = (B_1B_2 - B_3)B_3 - B_1^2 B_4 > 0.$$

Clearly $B_1 > 0$ and $B_3 > 0$ provided that:

$$y^* < \frac{1}{2} \quad (8c)$$

Straightforward computation shows that:

$$\Delta = [-\gamma_0\gamma_1(\gamma_0+\gamma_1) - \gamma_0(\gamma_3 - 2\gamma_2 - \gamma_5) + 2\gamma_1\gamma_5 + \gamma_6]B_3 - (\gamma_0+\gamma_1)^2 d_{11}\gamma_1 = l_1 - l_2 ,$$

$where$

$$l_1 = (\gamma_0+\gamma_1)[\gamma_0\gamma_1(2\gamma_0\gamma_2+\gamma_6) + (\gamma_3+\gamma_5)(\gamma_0\gamma_1^2 + \gamma_1(\gamma_3-\gamma_5))(\gamma_3-\gamma_5)(2\gamma_0\gamma_2 + \gamma_6)] \ , and$$

$$l_2 = \gamma_1(\gamma_3+\gamma_5)[4\gamma_0\gamma_2+\gamma_1(\gamma_3+\gamma_5)+2\gamma_6] + 2\gamma_0\gamma_2(\gamma_6+2\gamma_0\gamma_2)+\gamma_6(2\gamma_0\gamma_2+\gamma_6 + (\gamma_0+\gamma_1)^2 d_{11})$$

Hence, Δ will be positive if the following conditions hold:

$$l_1 > l_2 \quad (8d)$$

and ,

$$(r_1+r_2)(r_3+(1-m)w^* r_1(1-2\,y^*). \quad (8e)$$

Therefore, all the eigenvalues of J_3 have negative real parts under the given conditions and hence E_2 is locally asymptotically stable. However, it is unstable otherwise.

The global stability analysis of system(2)

In this section the global stability analysis for the equilibrium points, which are locally asymptotically stable, of system (2) is studied analytically with the help of Lyapunove method as shown in the following theorems.

***Theorem* (2):** Assume that the vanishing equilibrium point $E_0=(0,0,0,0)$ of system (2) is locally asymptotically stable in R_+^4. Then E_0 is globally asymptotically stable in R_+^4 .

Proof: Consider the following function:

$$V_0(x,y,z,w) = x+y+z+w,$$

Clearly V_0: $R_+^4 \to R$ is a C^1 positive definite function.

Now by differentiating V_0 with respect to time t and doing some algebraic manipulation, gives that:

$$\frac{dV_0}{dt} = y(1-y) - r_2 x - r_3 y - (1-r_4)(1-m)yw - r_5 z - r_7 w.$$

Now, due to the facts that is mentioned in proof of theorem (1), always $r_4<1$, we get,

$$\frac{dV_0}{dt} < y(1-r_3) - r_2 x - r_5 z - r_7 w \;.$$

Hence $\frac{dV_0}{dt}<0$ under condition (6b) and then V_0 is strictly Lyapunov function. Thus we obtain that E_0 is a globally asymptotically stable in R_+^4 and the proof is complete.

***Theorem* (3):** Assume that the free predators equilibrium point $E_1=(\bar{x},\bar{y},0,0)$ of system (2)is a locally asymptotically stable in R_+^4. Then E_1 is a globally asymptotically stable on the region $\omega_1 \subset R_+^4$ that satisfies the following conditions:

$$\frac{r_1}{y}+\frac{1}{x}-\frac{(y-\bar{y})}{x} \le 2\sqrt{\frac{r_1(y-\bar{y}^2)}{x\,\bar{x}y\bar{y}}} \quad (9\,a)$$

$$\bar{y} < (1-r_4)y \quad (9\,b)$$

Proof: Consider the following function

$$V_1(x,y,z,w) = c_1\left(x-\bar{x}-\bar{x}\ln\frac{x}{\bar{x}}\right)+c_2\left(y-\bar{y}-\bar{y}\ln\frac{y}{\bar{y}}\right)+c_3 z+c_4 w,$$

where : c_1 , c_2 , c_3 *and* c_4 are positive constants to be determined.

Clearly $V_1 : R_+^4 \to R$ is; a C^1 positive definite function. Now by differentiating V_1 with respect to time t and doing some algebraic manipulation, gives that:

$$\frac{dV_1}{dt} = -c_1\,\frac{(\bar{y}-\bar{y}^2)}{x\,\bar{x}}\,(x-\bar{x})^2 - \frac{r_1 c_2}{y\,\bar{y}}\,(y-\bar{y})^2 + \left[c_1\left(\frac{1}{x}-\frac{(y+\bar{y})}{x}\right)+\frac{c_2 r_1}{y}\right](x-\bar{x})(y-\bar{y}) - c_2(y-\bar{y})(1-m)w + c_3 r_4(1-m)yw - c_3 r_5 z - c_3 r_6 z + c_4 r_6 z - c_4 r_7 w.$$

By chosen $c_1=c_2=c_3=c_4=1$, we get:

$$\frac{dV_1}{dt} \le -\frac{(\bar{y}-\bar{y}^2)}{x\,\bar{x}}\,(x-\bar{x})^2 - \frac{r_1}{y\,\bar{y}}\,(y-\bar{y})^2 + \left[\left(\frac{1}{x}-\frac{(y+\bar{y})}{x}\right)+\frac{r_1}{y}\right](x-\bar{x})(y-\bar{y}) - (1-m)((1-r_4)y-\bar{y})w - r_5 z - r_7 w$$

Now by using the conditions(9 a) we obtain that:

$$\frac{dV_1}{dt} < -\left[\sqrt{\frac{(\bar{y}-\bar{y}^2)}{x\bar{x}}}\,(x-\bar{x}) - \sqrt{\frac{r_1}{y\bar{y}}}\,(y-\bar{y})\right]^2 - (1-m)\big((1-r_4)\,y-\bar{y}\big)\,w - r_5 z - r_7 w$$

Clearly, $\frac{dV_1}{dt}$ is negative definite on the region ω_1 due to the condition (9b). Hence V_1 is strictly Lyapunove function thus E_1 is a globally asymptotically stable on the region ω_1 and the proof is complete.

***Theorem* (4):** Assume that the positive (coexistence) equilibrium point $E_2=(x^*,y^*,z^*,w^*)$ of system (2) is locally asymptotically stable in the R_+^4 .Then E_2 is a globally asymptotically stable on any region $\omega_2 \subset R_+^4$ that satisfies the following conditions:

$$\frac{r_1}{y}+\frac{1}{x}-\frac{(y-y^*)}{x} \le 2\sqrt{\frac{(y^*-y^{*2})r_1}{2xx^*\;yy^*}} \quad (10\,a)$$

$$y^{*2} < y^* < y \quad (10\,b)$$

$$\frac{r_4}{z}(1-m)w \le \sqrt{\frac{r_1 r_4(1-m)}{yy^* zz^*}} \quad (10\,c)$$

$$\frac{r_4}{z}(1-m)y^*+\frac{r_6}{w} \le \sqrt{\frac{2r_4 r_6(1-m)}{zww^*}} \quad (10\,d)$$

Proof: Consider the following function:

$$V_2(x,y,z,w) = \left(x-x^*-x^*\ln\frac{x}{x^*}\right) + \left(y-y^*-y^*\ln\frac{y}{y^*}\right) + \left(z-z^*-z^*\ln\frac{z}{z^*}\right) + \left(w-w^*-w^*\ln\frac{w}{w^*}\right).$$

Clearly $V_2: R_+^4 \to R$ is a C^1 positive definite function. Now by differentiating $V_2$with respect to time t and doing some algebraic manipulation, gives that:

$$\frac{dV_2}{dt} = -\left(\frac{(y^*-y^{*2})}{x\,x^*}\right)(x-x^*)^2 - \frac{r_1}{y\,y^*}(y-y^*)^2 + \left[\left(\frac{r_1}{y}+\frac{1}{x}-\frac{(y-y^*)}{x}\right)\right](x-x^*)(y-y^*) - (1-m)(y-y^*)w - \frac{r_4}{zz^*}(1-m)(z-z^*)^2 + \frac{r_4}{z}(1-m)(z-z^*)(y-y^*)$$

$$+\frac{r_4}{z}(1-m)y^*\ (z-z^*)(w-w^*)$$
$$-\frac{r_6 z^*}{ww^*}(w-w^*)^2+\frac{r_6}{w}(w-w^*)(z-z^*).$$

So,

$$\frac{dV_2}{dt}\le-\left(\left(\left(\frac{(y^*-y^{*2})}{x\ x^*}\right)\right)(x-x^*)^2-\left(\frac{r_1}{y}+\frac{1}{x}-\frac{(y-y^*)}{x}\right)(x-x^*)(y-y^*)+\frac{r_1}{2y\ y^*}(y-y^*)^2\right)$$
$$-\left(\frac{r_1}{2y\,y^*}(y-y^*)^2\frac{r_4}{z}(1-m)y^*(z-z^*)(w-w^*)+\frac{r_4}{2zz^*}(1-m)(z-z^*)^2\right)$$
$$-\left(\frac{r_4}{2zz^*}(1-m)(z-z^*)^2\left(\frac{r_4}{z}(1-m)y^*+\frac{r_6}{w}\right)(w-w^*)(z-z^*)+\frac{r_6 z^*}{ww^*}(w-w^*)^2\right)$$
$$-(1-m)(y-y^*)w$$

Now by using the conditions$(10a)-(10d)$ we obtain that:

$$\frac{dV_2}{dt}<-\left[\sqrt{\frac{(y^*-y^{*^2})}{xx^*}}(x-x^*)-\sqrt{\frac{r_1}{2yy^*}}(y-y^*)\right]^2$$
$$-\left[\sqrt{\frac{r_1}{2yy^*}}(y-y^*)-\sqrt{\frac{r_4}{2zz^*}(1-m)}(z-z^*)\right]^2$$
$$-\left[\sqrt{\frac{r_4}{2zz^*}(1-m)}(z-z^*-\sqrt{\frac{r_6 z^*}{ww^*}}(w-w^*)\right]^2$$
$$-(1-m)(y-y^*).$$

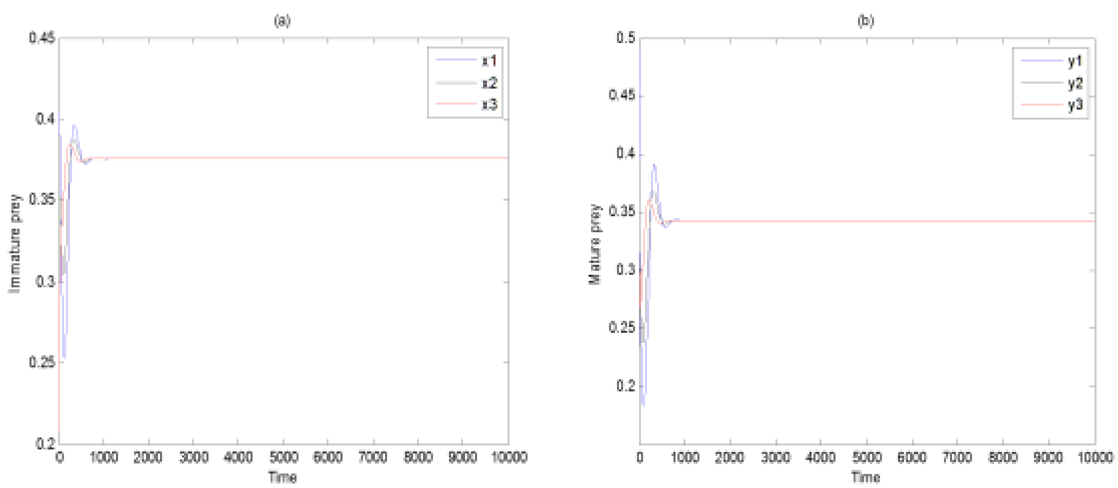
Clearly, $\frac{dV_2}{dt}$ is negative definite on the region ω_2 due to the conditions $(10b)$ .Hence V_2 is strictly Lyapunov function. Thus E_2 is a globally asymptotically stable on the region ω_2 and the proof is complete .

***Numerical analysis of system* (2)**

In this section, the dynamical behavior of system (2) is studied numerically for one set of parameters and three different initial points. The objectives of this study are: first investigate the effect of varying the value of each parameter on the dynamical behavior of system (2) and second confirm our obtained analytical results. It is observed that, for the following set of hypothetical parameters that satisfies stability conditions of the positive equilibrium point, system (2) has a globally asymptotically stable positive equilibrium point as shown in Fig.(11.1).

$$r_1=0.5\,,r_2=0.1\,,r_3=0.1\,,r_4=0.5\,,r_5=0.5\,,$$
$$r_6=0.5\,,r_7=0.1\,,m=0.3 \qquad (11)$$

Clearly, Fig. (11.1) shows that system (2) has a globally asymptotically stable as the solution of system (2) approaches asymptotically to the positive equilibrium point $E_2=(0.375\,,0.343\,,0.128\,,0.639)$ starting from three different initial points and this is confirming our obtained analytical results.



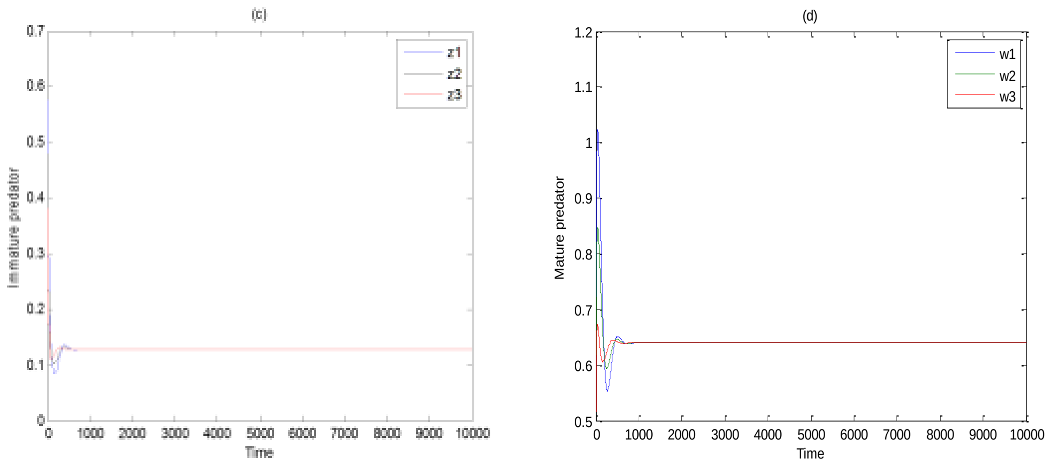


Fig.(11.1):-The time series of the solution of system (2) started from the three different initial points (0.4, 0.5, 0.6, 0.7), (0.3, 0.4, 0.5, 0.6), and (0.2, 0.3, 0.4, 0.5), for the data given by *eq*.(11) (a)- the trajectories of x as a function of time, (b) − the trajectories of y as a function of time, (c) trajectories of z as a function of time, (d) the trajectories of w as a function of time.

Now, in order to discuss the effect of the parameters values of system (2) on the dynamical behavior of the system, the system is solved numerically for the data given in (11) with varying one parameter at each time.

By varying the parameter r_1 which represents the conversion rate of the immature prey into mature prey and keeping the rest of parameters as data given in (11) in the range $0.1 \leq r_1 < 1$, it is observed that the solution of system (2) approaches asymptotically to the positive equilibrium point E_2, as shown in Fig.(11.2), for typical value $r_1 = 0.91$.

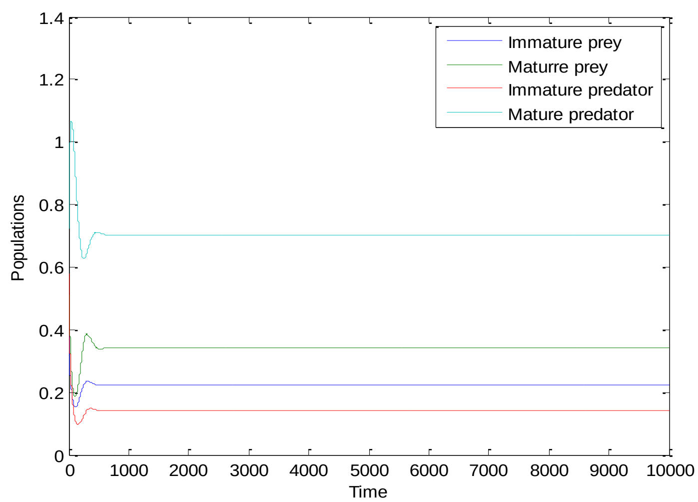
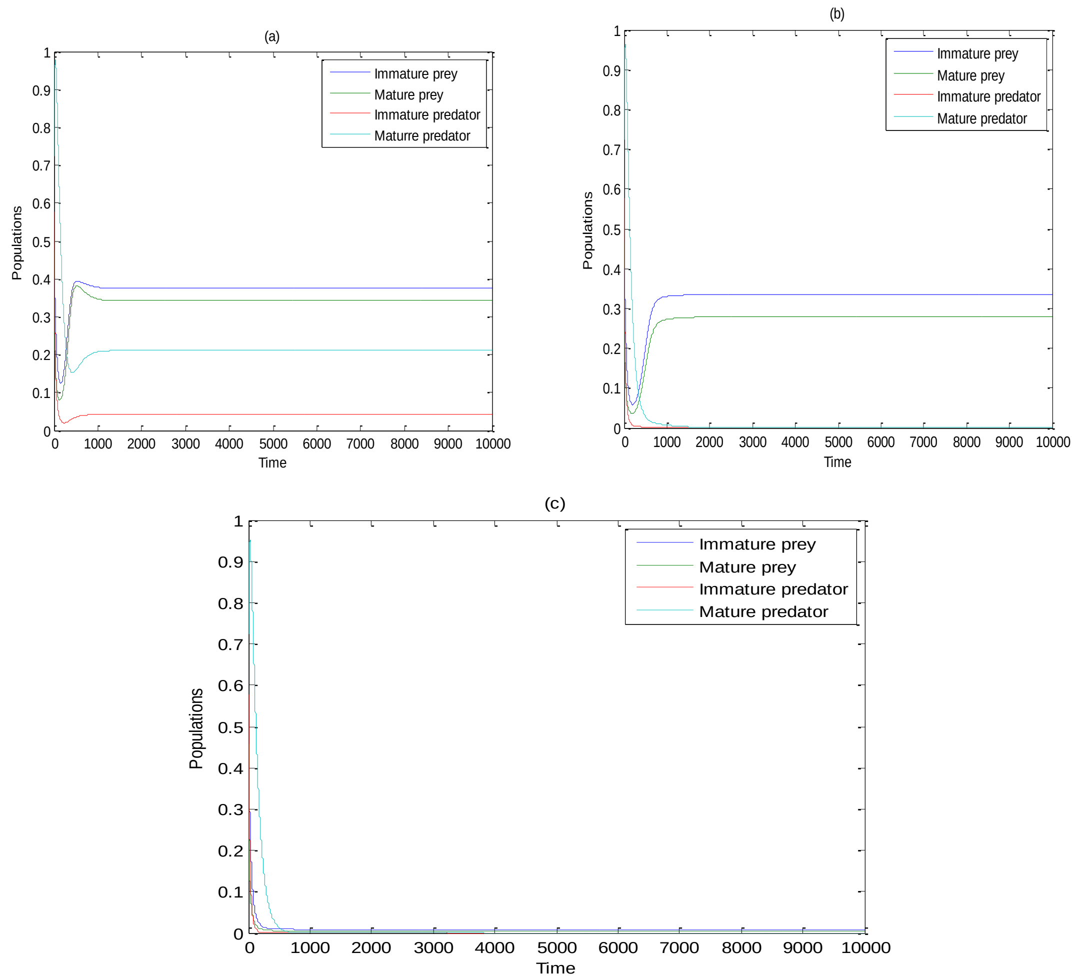


Fig.(11.2):-Time series of the solution of system (2) for the data given in (11) with $r_1 = 0.91$, which approaches to $E_2 = (0.22, 0.34, 0.14, 0.7)$ in the interior of R_+^4.

Now, varying the natural death rate parameter of immature prey r_2 and keeping the rest of parameters values as data given in (11), it is observed that for $0.01 \le r_2 < 1$ the solution of system (2) approaches asymptotically to a positive equilibrium point E_2.

On the other hand varying the natural death rate of mature predator parameter r_3 and keeping the rest of parameters values as data in (11), it is observed that for $0.01 \le r_3 < 0.55$ the solution of system (2) approaches asymptotically to the positive equilibrium point E_2, as shown in Fig. $(11.3)\,a$, for typical value $r_3 = 0.4$, while increasing this parameter for $0.55 \le r_3 < 0.829$ causes extinction in the predator and the solution of; system (2) approaches asymptotically to $E_1 = (\bar{x}, \bar{y}, 0, 0)$ in the interior of the positive quadrant of xy $-$plane, as; shown in Fig. $(11.3)\,b$, for typical value $r_3 = 0.6$. and increasing this parameter farther for 0.829≤ r_3<1 the solution of system (2) approaches; asymptotically to the vanishing equilibrium point E_0, as shown in Fig. (11.3) c, for typical value $r_3 = 0.83$.



Fig(11.3) (a) :Time series of the solution of system (2) for the data given by (11) with $r_3 = 0.4$, which approaches to $E_1 = (0.37, 0.34, 0.04, 0.21)$ in the interior of R^4_+, *and* (b): Time series of the solution of system (2) for the data given by (11) with $r_3 = 0.6$, which approaches to $E_1 = (0.34, 0.28, 0, 0)$ in the interior of the positive quadrant of xy $-$plane ,and (c) Time series of the solution of system (2) for the data given by (11) with $r_3 = 0.83$, which approaches to E_0 in the interior of R^4_+.

Moreover, varying the parameter; r_4 which represents the conversion rate of predation of the mature predator upon the mature prey, and keeping the rest of parameters values as data given in (11), it is observed that for $0.01 \le r_4 \le 0.19$ the solution of system (2) causes extinction in the predator and the solution of system (2) approaches asymptotically to the free predator equilibrium point E_1, in the interior of the positive quadrant of xy -plane as shown in Fig. (11.4) a, for typical value $r_4 = 0.19$, while increasing this parameter for $0.2 < r_4 < 1$ the solution of system (2) approaches asymptotically to the positive equilibrium point E_2 ,as shown in Fig. (11.4) b, for typical value $r_4 = 0.29$.

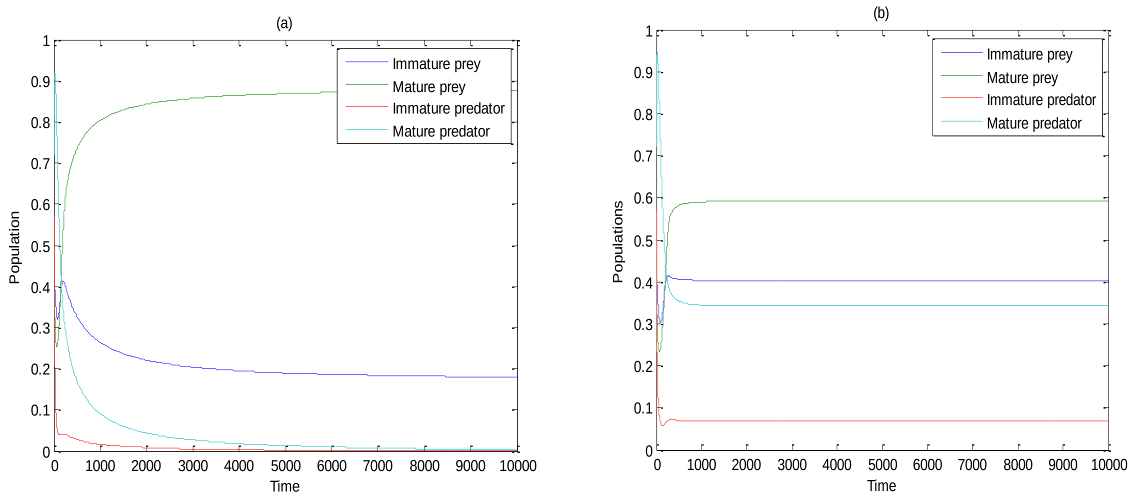


Fig . (11.4) (a) – Time series of the solution of system (2) for the data given by (11) with $r_4 = 0.19$, which approaches to $E_1 = (0.18, 0.88, 0, 0)$ in the interior of R^4_+ ,and (b) – Time series of the solution of system (2) for the data given by (11) with $r_4 = 0.29$, which approaches to $E_2 = (0.40, 0.59, 0.07, 0.34)$ in the interior of R^4_+ .

The varying of the parameters r_5 *and* r_6 which represents the natural death rate of the immature predator, and the conversion rate of the immature predator to mature predator respectively keeping the rest of parameters values as data given in (11), it is observed that for $0.01 \le r_5 < 1$ *and* $0.01 \le r_6 \le 1$ the solution of system (2) still approaches asymptotically to a positive equilibrium point E_2.

For varying the natural death rate parameter of the mature predator r_7, with $0.01 \le r_7 < 0.254$ the solution of system (2) approaches asymptotically to the; positive equilibrium point $E_2 = (x^*, y^*, z^*, w^*)$ in the interior of the interior of R^4_+, as shown in Fig.(11.5) a , for typical value $r_7 = 0.2$, while for $0.255 \le r_7 < 1$ the solution of system (2) approaches asymptotically to $E_1 = (\bar{x}, \bar{y}, 0,0)$ in the interior of the positive quadrant of $xy - plane$, as shown in Fig.(11.5) b , *for typical value* $r_7 = 0.5$.

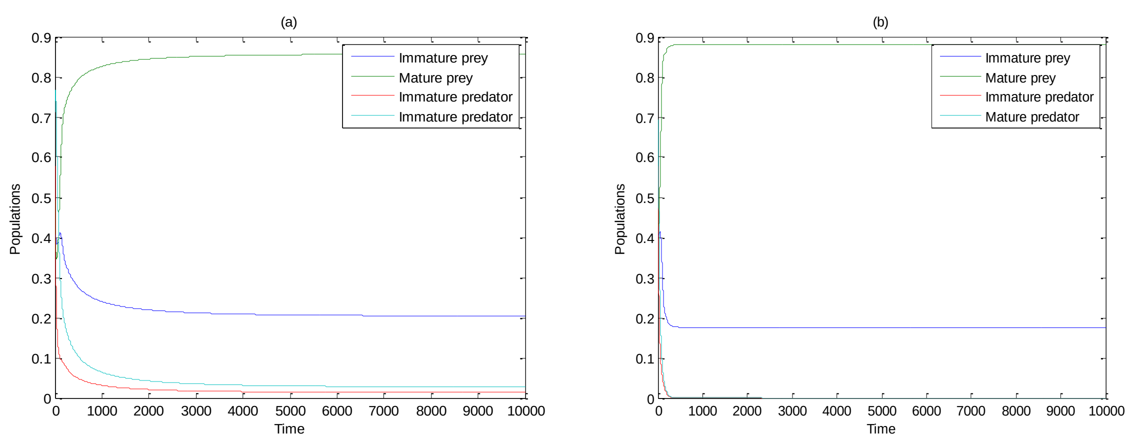
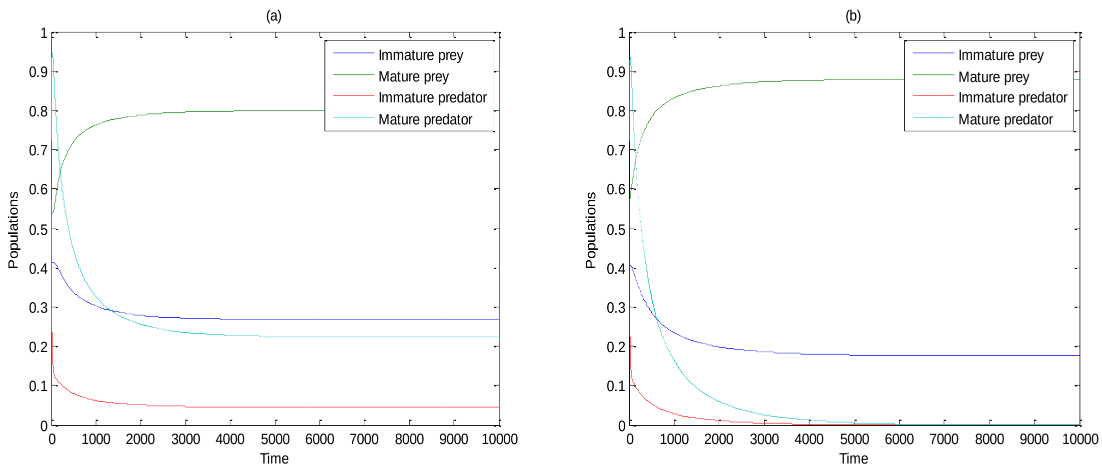


Fig .(11.5):- Time series of the solution of system (2) for the data given by (11) with different values of r_7 , (a): $E_2 = (0.20, 0.86, 0.014, 0.03)$ is a asymptotically stable with $r_7 = 0.2$, (b): $E_1 = (0.18, 0.88, 0, 0)$ is a asymptotically stable with $r_7 = 0.5$.

Finally, varying the number of prey inside the refuge parameter m and keeping the rest of parameters values as data given in (11), it is observed that for $0.01 \le m < 0.74$ the solution of system (2)approaches asymptotically to the positive equilibrium point E_2, as shown in Fig.(11.6) a, for typical value $m = 0.7$, while increasing this parameter in the range $0.74 \le m < 1$ leads that the solution of system (2) approaches asymptotically to free predator equilibrium point E_1 in Int. R^4_+, as shown in Fig. (11.6)b, for typical value $m = 0.75$.



Fig(11.6): – Time series of the solution of system (2) for the data given by (11) with different values of m, (a) - $E_2 = (0.27, 0.8, 0.04, 0.22)$ is a asymptotically stable with $m = 0.7$, (b) - $E_1 = (0.18, 0.88, 0, 0)$ is a asymptotically stable with m =0.75 .

CONCLUSIONS AND DISCUSSION:-

In this paper, we proposed and analyzed an ecological model that described the dynamical behavior of the prey-predator real system. The model included four non - linear autonomous differential equations that describe the dynamics of four different population, namely first immature prey (X), mature prey (Y), immature predator (Z) and (W) which is represent the mature predator. The boundedness of system (2) has been discussed. The existence conditions of all possible equilibrium points are obtain. The local as well as global stability analyses of these points are carried out. Finally, numerical simulation is used to specific the control set of parameters that affect the dynamics of the system and confirm our obtained analytical results. Therefore system (2) has been solved numerically for hypothetical set of parameter given in eq. (11) and different initial point and the following observations are obtained.

1- The system within the set of parameters imposed does not have a periodic solution.

2- For the set hypothetical parameters value given in eq. (11), the system (2) approaches asymptotically to globally stable positive point $E_2 = (0.375, 0.343, 0.128, 0.639)$. Further, with varying one parameter each time, it is observed that varying the parameter values, $r_i, i = 1,2,5 \ and \ 6$ do not have any effect on the dynamical behavior of system (2) and the solution of the system still approaches to positive equilibrium point $E_2 = (x^*, y^*, z^*, w^*)$.

3- As the natural death rate of mature prey r_3 increasing to 0.55 keeping the rest of parameters as in eq. (11), the solution of system (2) approaches to positive equilibrium point E_2 . However if $0.55 \leq r_3 < 0.829$, then the predator will face extinction and the trajectory transferred from positive equilibrium point to the equilibrium point $E_1 = (\bar{x}, \bar{y}, 0, 0)$, further more if $0.829 \leq r_3 < 1$ the solution of system (2) approaches to the vanishing equilibrium point E_0 thus, the parameters $r_3 = 0.55 \ and \ r_3 = 0.829$ are bifurcation points of system (2).

4- As the parameter r_4 which represents the conversion rate of food from the mature prey to the immature predator increasing to 0.19 keeping the rest of parameters as in eq.(11), the solution of system (2) approaches to the free predators equilibrium point $E_1 = (\bar{x}, \bar{y}, 0, 0)$, while for the values $0.19 < r_4 < 1$, then the trajectory transferred from the free predators equilibrium point $E_1 = (\bar{x}, \bar{y}, 0, 0)$ to $E_2 = (x^*, y^*, z^*, w^*)$, which means revival of the predator population thus, the parameter r_4 when $r_4 = 0.19$ is a bifurcation point of system (2).

5- As the natural death rate of the mature predator r_7 increasing to 0.254 keeping the rest of parameters as in eq.(11), the solution of system (2) approaches to the positive equilibrium point E_2, further increasing r_7 in the range $0.255 \leq r_7 < 1$ causes the predator faced extinction and the trajectory transferred from the positive equilibrium point E_2 to the free predator equilibrium point $E_1 = (\bar{x}, \bar{y}, 0, 0)$, thus, the parameter r_7 when $r_7 = 0.255$ is a bifurcation point of system (2).

6- As the number of prey inside the refuge m varying in the range $0.01 \leq m < 0.74$ and keeping the rest of parameters values as data given in eq.(11), the solution of system (2) approaches asymptotically to the positive equilibrium point E_2, while increasing this parameter in the range $0.74 \leq m < 1$ causes extinction of the predator population and the trajectory transferred from $E_2 = (x^*, y^*, z^*, w^*)$, to $E_1 = (\bar{x}, \bar{y}, 0, 0)$, thus, the parameter m when $m = 0.74$ is a bifurcation point of system (2).

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