HESITANT FUZZY ABEL-GRASSMANN'S GROUPOIDS

Muhammad Gulistan[1,a], Muhammad Shahzad[2] , Farooq Ahmad[3], Muhammad Azam[4], Shah Nawaz [5]
gulistanmath@hu.edu.pk[1], shahzadmaths@hu.edu.pk[2], f.ishaq@mu.edu.sa[3],
muhammadazammath@gmail.com[4], shahnawazawan82@gmail.com[5]
[1,2,4,5]Department of Mathematics, Hazara University, Mansehra, Pakistan
[3]Presently Mathematics Department, Majmaah University, College of Science, Alzulfi, KSA
[3]Punjab Higher Education Department, College wing, Lahore, Pakistan
a: (Corresponding author) gulistanmath@hu.edu.pk

ABSTRACT: *In this paper we give the idea of hesitant fuzzy ideals, hesitant fuzzy bi-ideals, hesitant fuzzy interior ideals, hesitant fuzzy quasi ideals of an AG-groupoid. We also discuss the inter relationships between the hesitant fuzzy ideals. Further we show that if H is a hesitant fuzzy sub AG-groupoid, then H is a hesitant fuzzy bi-ideal if $(H \circ G \circ H) \subseteq H$ and is interior ideal if $(G \circ H \circ G) \subseteq H$. Moreover we show that in an AG-groupoid a fuzzy hesitant fuzzy ideal is a hesitant fuzzy quasi-ideal but the converse is not true.*

2010 Mathematics Subject Classifications: 20M10, 20N99.

Key Words: AG-groupoid, fuzzy ideals, (quasi, bi, generalized bi, interior) ideals, hesitant fuzzy left(right) ideals, hesitant fuzzy (qusi, bi, generalized, interior) ideals.

1. INTRODUCTION

An Abel Grassmann's groupoid, abbreviated as AG-groupoid, is a groupoid G whose elements satisfy the law, $(ab)(cd)=(ac)(bd)$ for all $a,b,c,d \in G$ holds [9,10]. left invertive law: $(ab)c=(cb)a$ for all $a,b,c \in G.$ An AG-groupoid is the midway structure between a commutative semigroup and a groupoid. There are several authors who explored this idea further and added many usefull results to the theory of AG-groupoids, for instance, Mushtaq et al. [19-20], Gulistan et al. [6], Khan et al. [11-12,17] and Yaqoob et al. [22, 26].

The theory of a fuzzy subset of a set was first time discovered by Zadeh [31]. The theory of fuzy sets have been applied to algebras by several researchers, for instance, Akram et al. [1], Aslam et al. [2], Faisal et al. [3,4,5], Gulistan et al. [7], Khan et al. [13-16,], Yaqoob et al. [23-25, 27-29] and Yousafzai et al. [30].

Torra [21] introduced the notion of hesitant fuzzy sets. Let X be a given set, a hesitant fuzzy set abbreviated by (HFSs) can be defined in the term of a function that when applied to a set X again returns a subset of $[0,1]$. Hesitant fuzzy set is a usefull generalization of the fuzzy set. The hesitant fuzzy set permits the membership degree of an element to a set to be represented by a set of possible values between 0 and 1. Similar to the situations of hesitant fuzzy set where a decision maker may hesitate between several possible values as the membership degree when evaluating an alternative, in a qualitative circumstance, a decision maker may hesitate between several terms to assess a linguistic variable. Hesitant fuzzy set theory has been applied to many practical problems, primarily in the area of decision making. Jun and Song [8] applied the theory of Hesitant fuzzy sets to MTL-algebras.

In this paper we define hesitant fuzzy bi-ideal, generalized hesitant fuzzy bi-ideals, hesitant fuzzy interior ideal, and hesitant fuzzy quasi-ideals. We show that if H is a hesitant fuzzy sub AG-groupoid, then H is a hesitant fuzzy bi-ideal if $(H \circ G \circ H) \subseteq H$ and is interior ideal if $(G \circ H \circ G) \subseteq H$. Further we show that if H is a hesitant fuzzy sub AG-groupoid of G, then H being a quasi-ideal if $(H \circ G) \bigcap (G \circ H) \subseteq H.$ Moreover we show that in an AG-groupoid a fuzzy hesitant fuzzy ideal is a hesitant fuzzy quasi-ideal but the converse is not true.

2. PRELIMINARIES

In this section we give some basic definition and results, which will be used in our main section.

Definition 2.1: [18] A non-empty subset λ of G is called left(right) ideal of an AG-groupoid G if for all $G \circ \lambda \subseteq \lambda (\lambda \circ G \subseteq \lambda).$ It is called an ideal of AG-groupoid if it is both left and right ideal of λ .

Definition 2.2: [18] A non-empty subset λ of an AG-groupoid G is said to be a generalized bi-ideal of G if $(\lambda \circ G) \circ \lambda \subseteq \lambda$ if λ is a sub AG-groupoid of G then G is said to be a bi-ideal of G if, $(\lambda \circ G) \circ \lambda \subseteq \lambda.$

Definition 2.3: [18] A non-empty subset λ of an AG-groupoid G is said to be a quasi-ideal of G if $(G \circ \lambda) \bigcap (\lambda \circ G) \subseteq \lambda$.

Definition 2.4: [18] A non-empty subset λ of an AG-groupoid G is said to be an interior ideal if $(G \circ \lambda) \circ G \subseteq \lambda.$

Lemma 2.5: [18] Let λ_1 and λ_2 be two left (right, two sided ideal) of an AG-groupoid G . Then the product of λ_1 and λ_2 is a left (right, two sided ideal) of G .

Definition 2.6: [31] A fuzzy subset β of a set X is a function of X into the closed interval [0,1] that is $\beta : X \to [0,1].$

Definition 2.7: [32] A fuzzy subset η of an AG-groupoid G is a function of G into the closed unit interval [0,1] that is $\beta : G \to [0,1].$

Definition 2.8: [32] In an AG-groupoid G, a fuzzy subset η of G is called a fuzzy sub AG-groupoid if $\beta(xy) \geq \beta(x) \wedge \beta(y),\ \forall\ x, y \in G.$

Definition 2.9: [32] Let β be fuzzy subset of an AG-groupoid G. Then β is called fuzzy left ideal and fuzzy right ideal of G if $\beta(xy) \geq \beta(y)$ for all x,y∈G. And $\beta(xy) \geq \beta(x)$ for all $x, y \in G$ and is called fuzzy two-sided ideal of G if it is both a fuzzy left and fuzzy right ideal of G.

Theorem: [32] Let G be an AG-groupoid with left identity e. Then evry fuzzy right ideal of G is a fuzzy left ideal of G.

Definition 2.10: A fuzzy subset β of an AG-groupoid G is called a fuzzy quasi-ideal of G if $(\beta \circ G)\bigcap(G \circ \beta) \subseteq \beta.$ A fuzzy subset β of an AG-groupoid G is called a fuzzy generalized bi-ideal of G if $\beta((ta)z) \supseteq \beta(t) \bigcap \beta(z).$ for all $a, t, z \in G$. A fuzzy subset β of an AG-groupoid G is called fuzzy interior ideal of G if $\beta((ta)z) \supseteq \beta(a)$ for all $a, t, z \in G$.

Definition 2.11: [18] Let β_1 and β_2 be two fuzzy subsets of an AG-groupoid G. Then $\beta_1 \circ \beta_2$ is defined as:

$$(\beta_1 \circ \beta_2)(g) = \begin{cases} \bigcup\limits_{\text{g=ab}} \{\beta_1(a) \bigcap \beta_2(b)\} & if\ \ g = ab,\ \forall\ a, b, g \in G, \\ 0 & otherwise. \end{cases}$$

Definition 2.12: [18] Let β_1 and β_2 be two fuzzy subsets of G. Then the intersection and union of β_1 and β_2 can be defined as follow:

$$(\beta_1 \bigcap \beta_2)(t) = \ min\{\beta_1(t), \beta_2(t)\}, \\ = \ \beta_1(t) \wedge \beta_2(t)$$

and

$$(\beta_1 \bigcup \beta_2)(t) = \ max\{\beta_1(t), \beta_2(t)\}, \\ = \beta_1(t) \vee \beta_2(t), \forall\ t \in G.$$

3. HESITANT ABEL-GRASSMANN'S GROUPOIDS

In this section we define hesitant fuzzy ideals, hesitant fuzzy bi-ideals, hesitant fuzzy interior ideals and hesitant fuzzy quasi ideals in AG-groupoid with examples. Further we will study some properties.

Definition 3.1: Let H be a hesitant fuzzy set of AG-groupoid G. Then we have

$$H_a = H(a),\ \mathrm{H}_a^b = H(a) \cap H(b),\ \mathrm{H}_a^b[c] \\ = H(a) \cap H(b) \cap H(c).$$

So we get that $H_a^b = H_b^a$

$$H_a = H_b \Leftrightarrow H_a \subseteq H_b,\quad H_b \subseteq H_a,$$

for all $a, b \in G.$

Definition 3.2: Let H_1 and H_2 be two hesitant fuzzy sets of G. Then hesitant union $H_1 \cup H_2$ and hesitant intersection $H_1 \cap H_2$ of H_1 and H_2 are define to be hesitant fuzzy sets of G as follow:

$$H_1 \cup H_2 : \mathrm{G} \to P([0,1]), \\ \mathrm{a} \to H_{1a} \cup H_{2a} = \max\{H_{1a},\ H_{2a}\}$$

and

$$H_1 \cap H_2 : \mathrm{G} \to P([0,1]), \\ \mathrm{a} \to H_{1a} \cap H_{2a} = \min\{H_{1a},\ H_{2a}\}.$$

Definition 3.3: A hesitant fuzzy set H on AG-groupoid G is called a hesitant fuzzy sub AG-groupoid of G if it satisfies:

$$H_{ab} \supseteq H_a^b = H_a \cap H_b,\ \forall\ a, b \in G.$$

Definition 3.4: A hesitant fuzzy set H of G is called a hesitant fuzzy left (right) ideal of G if it satisfies:

$$H_{ab} \supseteq H_a (H_{ab} \supseteq H_b),\ \forall\ a, b \in G.$$

Definition 3.5: A hesitant fuzzy left ideal and a hesitant fuzzy right ideal of an AG-groupoid ***G*** is called hesitant fuzzy two-sided ideal of G.

In the following we give some examples of hesitant fuzzay set of AG-groupoid G.

Example 3.6: Let $G = \{a, b, c\}$ be an AG-groupoid whith the binary operation define in the table

∘	a	b	c
a	b	c	b
b	b	b	b
c	b	b	b

Let H be a hesitant fuzzy set on G defined as follows:

$$H : G \to P([0,1]),\quad \mathrm{t} \to \begin{cases} [0,0.2) & \text{if t=a,} \\ [0.1,0.5] & \text{if t=b,} \\ [0.1,0.5] \cup (0,0.3) & \text{if t=c.} \end{cases}$$

Then H is a hesitant fuzzy left (resp., right) ideal of G.

Remarks 3.7: Every hesitant fuzzy left (right) ideal of G is a hesitant fuzzy sub AG-groupoid of G. But the converse is not true as show in the following example.

Example 3.8: Let $G = \{a, b, c\}$ be an AG-groupoid and let H be a hesitant fuzzy set of G define in the example and define the hesitant fuzzy set of G as follow:

$$H: G \to P([0,1]),\ t \mapsto \begin{cases} [0.3,0.4) & \text{if t=a,} \\ [0.1,1.5] & \text{if t=b,} \\ (0,0.3)\cap[0.1,0.5] & \text{if t=c.} \end{cases}$$

Then H is a hesitant fuzzy sub AG-groupoid of G, but it is not a hesitant fuzzy ideal of G, because

$$H_{ab} \supseteq H_b \Rightarrow H_c \times H_b.$$

Definition 3.9: Let H_1 and H_2 be two hesitant fuzzy sets of G. Then their product defined as follows:

$$(H_1 \circ H_2)_a = \begin{cases} \bigcup_{a=bc} \{H_{1b} \cap H_{1c}\} \\ \phi \quad otherwise. \end{cases}$$

In the following we give some basic properties of a hesitant fuzzy set of an AG-groupoid.

Proposition 3.10: If G is an AG-groupoied, then the collection of all hesitant fuzzy sets $(HF(G), o)$ is an AG-groupoid.

Proof: Since $HF(G)$ is closed. Let $H \in HF(G)$. Then $HoH \subseteq HF(G)$, such that

$$(HoH)_a = \bigcup_{a=bc}\{H_b \cap H_c\} \subseteq \bigcup_{a=bc} H_b{}^c = H_a \in HF(\text{G}).$$

Thus GF(H) is closed.

Let $H_1, H_2, H_3 \in HF(G)$.

Then we have,

$$((\text{H}_1\text{oH}_2)\text{oH}_3)_t = \bigcup_{t=yz}\{(\text{H}_1\text{oH}_2)_y \cap \text{H}_{3z}\}$$
$$= \bigcup_{t=yz}\left\{\bigcup_{y=pq}\{\text{H}_{1p} \cap \text{H}_{2q}\} \cap H_{3z}\right\}$$
$$= \bigcup_{t=(pq)z}\{(H_{1p} \cap H_{2q}) \cap H_{3z}\}$$
$$= \bigcup_{t=(zq)p}\{(H_{3z} \cap H_{2q}) \cap H_{1p}\}$$
$$= \bigcup_{t=wz}\{(H_3\text{o}H_2)_w \cap H_{1p}\}$$
$$= ((\text{H}_3\text{oH}_2)\text{oH}_1)$$
$$\Rightarrow (\text{H}_1\text{oH}_2)\text{oH}_3 = (\text{H}_3\text{oH}_2)\text{oH}_1.$$

Hence $(HF(G), o)$ is a AG-groupoid.

Corollary 3.11: If G is an AG-groupoid , then the medial law holds in $HF(G)$.

Proof: Let $H_1, H_2, H_3, H_4 \in HF(G)$. Then we have:

$$((H_1\text{o}H_2)\text{o}(H_3\text{o}H_4))_t = \bigcup_{t=ab}\{(H_1\text{o}H_2)_a \cap (H_3\text{o}H_4)_b\}$$
$$= \bigcup_{t=ab}\left\{\bigcup_{a=cd}\{H_{1c}\text{o}H_{2d}\} \cap \bigcup_{b=ef}\{H_{3e}\text{o}H_{4f}\}\right\}$$
$$= \bigcup_{t=(cd)(ef)}\{(H_{1c}\text{o}H_{2d}) \cap (H_{3e} \cap H_{4f})\}$$
$$= \bigcup_{t=(ce)(df)}\{(H_{1c}\text{o}H_{3e}) \cap (H_{4f} \cap H_{3d})\}$$
$$= \bigcup_{t=wz}\left\{\bigcup_{w=ce}\{H_{1c}\text{o}H_{3e}\} \cap \bigcup_{z=df}\{H_{2d}\text{o}H_{4f}\}\right\}$$
$$= \bigcup_{t=wz}\{(H_1\text{o}H_3)_w \cap (H_2\text{o}H_4)_z\}$$
$$= ((H_1\text{o}H_3)\text{o}(H_2\text{o}H_4))_t$$

Thus,

$$(H_1\text{o}H_2)\text{o}(H_3\text{o}H_4) = (H_1\text{o}H_3)\text{o}(H_2\text{o}H_4).$$

Theorem 3.12: If G be an AG-groupoid with left identity e, then the two properties hold in $HF(G)$:

(a) $\quad H_1o(H_2oH_3) = H_2o(H_1oH_3).$

(b) $\quad (H_1oH_2)o(H_3oH_4) = (H_4oH_3)o(H_2oH_1),$

$\forall\ H_1, H_2, H_3, H_4 \in F(G).$

Proof: (a) Let $t \in G,\ s.t,\ t \neq yz,\ \forall\ \text{y,z} \in \text{G}.$ Then

$$(H_1\text{o}(H_2\text{o}H_3))_t = \phi = (H_2\text{o}(H_1\text{o}H_3))_t.$$

If $t = yz,\ \forall\ t, y, z \in G,$ then

$$(H_1\text{o}(H_2\text{o}H_3))_t = \bigcup_{t=yz}\{H_{1y} \cap (\text{H}_2\text{oH}_3)_z\}$$
$$= \bigcup_{\text{t=yz}}\left\{H \cap \bigcup_{z=pq}\{\text{H}_{2p}\text{H}_{3q}\}\right\}$$
$$= \bigcup_{t=y(pq)}\{(\text{H}_{1y} \cap H_{2\text{p}}) \cap \text{H}_{3\text{q}}\}$$
$$= \bigcup_{t=y(pq)}\{(\text{H}_{2p} \cap H_{1\text{y}}) \cap \text{H}_{3\text{q}}\}$$
$$= \bigcup_{t=y(pq)}\left\{\text{H}_{2p} \cap \bigcup_{w=yq}\{H_{1y} \cap H_{3q}\}\right\}$$
$$= \bigcup_{t=y(pq)}\{\text{H}_{2p} \cap (H_1 \circ H_2)_w\}$$
$$= (H_2\text{o}(H_1\text{o}H_3))_t$$

Thus

$$(H_1 \circ (H_2 \circ H_3) = (H_2 \circ (H_1 \circ H_3).$$

(b): $If\ \ t \in G,\ s.t,\ t \neq yz,\ \forall\ y, z \in G,$

Then

$$((H_1 \circ H_2) \circ (H_3 \circ H_4))_t = \phi = ((H_4 \circ H_3) \circ (H_2 \circ H_1))_t.$$

$If\ \ t = yz,\ then$

$$\left((H_1 o H_2) o (H_3 o H_4)\right)_t = \bigcup_{t=yz}\left\{(H_1 \circ H_2)_y \cap (H_3 o H_4)_z\right\}$$

$$=\bigcup_{t=yz}\left\{\bigcup_{y=pq}\left\{H_{1p}\cap H_{2q}\right)\right\}\cap\bigcup_{z=uv}\left\{H_{3u}\cap H_{4v}\right\}\right\}$$

$$=\bigcup_{t=(pq)(uv)}\left\{\left(H_{1p}\cap H_{2q}\right)\cap\left(H_{3u}\cap H_{4v}\right)\right\}$$

$$=\bigcup_{t=(vu)(qp)}\left\{\left(H_{4v}\cap H_{3u}\right)\cap\left(H_{2q}\cap H_{1p}\right)\right\}$$

$$=\bigcup_{t=mn}\left\{\bigcup_{m=pq}\left\{H_{4v}\cap H_{3u}\right)\right\}\cap\bigcup_{n=qp}\left\{H_{2q}\cap H_{1p}\right\}\right\}$$

$$=\bigcup_{t=mn}\left\{(H_4\circ H_3)_m\cap(H_2 o H_1)_n\right\}$$

$$=\left((H_4 o H_3) o (H_2 o H_1)\right)_t$$

$$\Rightarrow\left((H_1 o H_2) o (H_3 o H_4)\right)_t=\left((H_4 o H_3) o (H_2 o H_1)\right)_t.$$

Hence proof.

Proposition 3.13: An AG-groupoid G with $HF(G)=\left(HF(G)\right)^2$ is commutative semigroup if and only if $(H_1 o H_2) o H_3 = H_1 o (H_3 o H_2)$.

Proof: Suppose G is commutative semigroup. For any hesitant fuzzy subsets H_1,H_2 and H_3 of G by use Preposition P1, and commutative law:

$$((H_1 o H_2) o H_3)_t=\bigcup_{t=ab}\left\{(H_1 o H_2)_a\cap H_{3b}\right\}$$

$$=\bigcup_{t=ab}\left\{\bigcup_{a=cd}\left\{H_{1c}\cap H_{2d}\right\}\cap H_{3b}\right\}$$

$$=\bigcup_{t=c(db)}\left\{H_{1c}\cap\left(H_{2d}\right)\cap H_{3b}\right\}$$

$$=\bigcup_{t=c(bd)}\left\{H_{1c}\cap\left(H_{3b}\right)\cap H_{2d}\right\}$$

$$=\bigcup_{t=cw}\left\{H_{1c}\cap\bigcup_{w=bd}\left\{H_{3b}\cap H_{2d}\right\}\right\}$$

$$=\bigcup_{t=cw}\left\{H_{1c}\cap(H_3\circ H_2)_w\right\}$$

$$=(H_1 o (H_3 o H_2))_t.$$

Thus,

$$(H_1 o H_2) o H_3 = H_1 o (H_3 o H_2)$$

Conversely, suppose that,

$$(H_1 o H_2) o H_3 = H_1 o (H_3 o H_2)$$

holds for all fuzzy subsets $H_1, H_2, H_3 \in G.$ Now we want to show an AG-groupoid G is a commutative semigroup. Let H_1 and H_2 be any two arbitrary hesitant fuzzy subsets of G . Since $HF(G)=\left(HF(G)\right)^2$. *So* $H_1 = H_3 o H_4,$ where H_3 and H_4 are any hesitant fuzzy subsets of G . Now

$$(H_1 o H_2)_t=((H_3 o H_4) o H_2)_t=\bigcup_{t=ab}\left\{(H_3 o H_4)_a\cap H_{2b}\right\}$$

$$=\bigcup_{t=ab}\left\{\bigcup_{a=cd}\left\{H_{3c} o H_{4d}\right\}\cap H_{2b}\right\}$$

$$=\bigcup_{t=c(db)}\left\{\left(H_{3c}\cap H_{4d}\right)\cap H_{2b}\right\}$$

$$=\bigcup_{t=b(cd)}\left\{\left(H_{2b}\cap H_{3c}\right)\cap H_{4d}\right\}$$

$$=\bigcup_{t=bz}\left\{H_{2b}\cap\bigcup_{z=cd}\left\{H_{3c}\cap H_{4d}\right\}\right\}$$

$$=\bigcup_{t=bz}\left\{H_{2b}\cap(H_3\circ H_4)_z\right\}=\bigcup_{t=bz}\left\{H_{2b}\cap H_{1z}\right\}$$

$$=(H_2\circ H_1)_t$$

Thus

$$H_1 o H_2 = H_2 o H_1,$$

Which show that commutative law holds in G. Now by using Proposition1 and commutative law.

4. HESITANT FUZZY BI-IDEALS IN AG-GROUPOIDS

Here we define hesitant fuzzy Bi-ideals in AG-groupoid. We give some examples and some properties.

Definition 3.14: A hesitant fuzzy subset H of an AG-groupoid G is called a hesitant fuzzy bi-ideal of G if

$$H_{(xy)z}\supseteq H_x\cap H_z,\ \textit{for all } x,y,z\in G.$$

Example 3.15: G={a,b,c} be an AG-groupoid with binary operation define in table T1:

$$H: G\to p([0,1]),\ x\mapsto\begin{cases}[0.2,0.6) & \text{if x=a,}\\ [0.2,0.5] & \text{if x=b,}\\ (0.1,0.2)\cap[0.3,0.7) & \text{if x=c.}\end{cases}$$

Then H is a hesitant fuzzy bi-ideal on G.

Definition 3.16: A non-empty subset A of an AG-groupoid G is called the characteristic function of G if,

$$[H_{\mathrm{A}}]: G\to P([0,1]),\ t\mapsto\begin{cases}[0,1] & \textit{if } t\in\mathrm{A},\\ \phi & \textit{otherwise.}\end{cases}$$

Theorem 3.17: For a non-empty subset A of an AG-groupoid G is called a bi-ideal of G if and only if the characteristic function $[H_{\mathrm{A}}]$ of an AG-groupoid G is a hesitant fuzzy bi-ideal of G.

Proof: Suppose that A is a bi-ideal of an AG-groupoid G.

Let $a,b,c\in G,$ if $a,c\notin\mathrm{A},$ then

$$[H_{\mathrm{A}}]_a\cap[H_{\mathrm{A}}]_c=\phi\subseteq[H_{\mathrm{A}}]_{(ab)c}$$

$$\Rightarrow[H_{\mathrm{A}}]_a\cap[H_{\mathrm{A}}]_c\subseteq[H_{\mathrm{A}}]_{(ab)c}.$$

If $a,c\in \mathrm{A},$ then $ac\in \mathrm{A}.$ Since A is a bi-ideal of $G.$ Then

$$[H_{\mathrm{A}}]_{(ab)c}=[0,1]=[H_{\mathrm{A}}]_a\cap[H_{\mathrm{A}}]_c.$$

Thus $[H_{\mathrm{A}}]$ is a hesitant fuzzy bi-ideal of $G.$

Conversely, assume that $[H_{\mathrm{A}}]$ is a hesitant fuzzy bi-ideal of $G.$ Let $a,c\in \mathrm{A}$ and $b\in G$ then

$$[H_{\mathrm{A}}]_a\cap[H_{\mathrm{A}}]_c=[0,1]$$

and thus

$$[H_{\mathrm{A}}]_{(ab)c}\supseteq[H_{\mathrm{A}}]_a\cap[H_{\mathrm{A}}]_c=[0,1]$$
$$\Rightarrow[H_{\mathrm{A}}]_{(ab)c}=[0,1].$$

Which show that $(ab)c\in \mathrm{A}G\mathrm{A}.$ Thus A is a bi-ideal of $G.$

Lemma 3.16: Let H be a hesitant fuzzy sub AG-groupoid of an AG-groupoid G . Then H is a hesitant fuzzy bi-ideal of G if and only if

$$(HoG)oH\subseteq H.$$

Proof: Suppose that, H be a hesitant fuzzy bi-ideal of G . Let $x=ab,\ for\ all\ ,a,b\in G.$ Then

$$(HoGoH)_x=\bigcup_{x=ab}\{(HoG)_a\cap H_b\}=\bigcup_{x=ab}\left\{\bigcup_{a=pq}\{H_p\cap G_q\}\cap H_b\right\}$$
$$=\bigcup_{x=ab}\left\{\bigcup_{a=pq}\{H_p\cap[0,1]\}\cap H_b\right\}=\bigcup_{x=(pq)b}\{H_p\cap H_b\}$$
$$\subseteq\bigcup_{x=(pq)b}\{H_{(pq)b}\}$$
$$\subseteq H_{x.}$$

Thus, $HoGoH\subseteq H.$

Conversly, assume that: $HoGoH\subseteq H.$

Let $x=(ab)c,\ for\ all\ a,b,c,x\in G.$

$$H_x\supseteq(HoGoH)_x=\bigcup_{x=pq}\{(HoG)_p\cap H_q\}$$
$$=\bigcup_{(ab)c=pq}\{(HoG)_p\cap H_q\}$$
$$\supseteq(HoG)_{ab}\cap H_c$$
$$=\bigcup_{(ab)=uv}\{H_u\cap G_v\}\cap H_c$$
$$\supseteq(H_a\cap G_b)\cap H_c$$
$$=(H_a\cap[0,1]\cap H_c$$
$$=H_a\cap H_c.$$

Thus, $H_{(ab)c}\supseteq H_a\cap H_c.$

5. HESITANT FUZZY INTERIOR IDEALS IN AG-GROUPOIDS

Definition 3.17: A hesitant fuzzy subset H of an AG-groupoid G is called a hesitant fuzzy interior ideal of G if $H_{(xa)y}\supseteq H_a,\ for\ all\ a,b,c\in G.$

Example 3.18: Let $G=\{a,b,c\}$ be an AG-groupoid with the binary operation definein table $T1.$

$$H:G\to p([0,1]),\ x\mapsto\begin{cases}[0.3,0.9) & \text{if x=a,}\\ [0.1,0.7] & \text{if x=b,}\\ (0.3,0.8] & \text{if x=c.}\end{cases}$$

Then H is a hesitant fuzzy interior ideal of $G.$

Definition 3.19: Let H be a hesitant fuzy set of an AG-groupoid G and $\varepsilon\subseteq[0,1],$ then $G(H;\varepsilon)=\{t\in G/\varepsilon\subseteq H_t\},$ is called the hesitant level set of H .

Theorem 3.20: A hesitant fuzzy interior ideal H of an AG-groupoid G is a hesitant fuzzy sub AG-groupoid of G if and only if the set $G(H;\varepsilon)=\{t\in G/\varepsilon\subseteq H_t\}$ is a sub AG-groupoid, when $\varepsilon\in P([0,1])$.

Proof: Let H be a hesitant fuzzy interior ideal of $G.$ Let $x,y,z\in G(H;\varepsilon)$. Implies that $\varepsilon\in H_{xy}$ and $\varepsilon\in H_z,$ but by hypothesis $\varepsilon\subseteq H_{(xy)z}\Rightarrow(xy)z\in G(H;\varepsilon).$ Which show that $G(H;\varepsilon)$ is a sub AG-groupoid of G .

Conversly, assume that $G(H;\varepsilon)$ is a sub AG-groupoid of $G.$ Let $x,y,z\in G$ such that $H_{(xy)z}\supseteq\varepsilon\supseteq H_y,$ then $x,y,z\in G(H;\varepsilon),$ but $(xy)z\notin G(H;\varepsilon),$ which show that $H_{(xy)z}\supseteq H_y.$

Lemma 3.19: Let H be a hesitant fuzzy sub AG-groupoid of an AG-groupoid G . Then H is a hesitant fuzzy interior ideal of G if and only if GoHoG⊑H.

Proof: Let H be a hesitant fuzzy interior ideal of G. Then

$$H_{(xa)y}\supseteq H_a.$$

Let a,b,c∈G

$$(GoHoG)_x=\bigcup_{x=yz}\{(GoH)_y\cap G_z\}=\bigcup_{x=yz}\left\{\bigcup_{y=ab}\{G_a\cap H_b\}\cap G_z\right\}$$
$$=\bigcup_{x=yz}\left\{\bigcup_{y=aob}\{[o,1]\cap H_b\}\cap[0,1]\right\}=\bigcup_{x=yz}\left\{\bigcup_{y=ab}\{H_b\}\cap[0,1]\right\}$$
$$=\bigcup_{x=(ab)z}\{H_b\}\subseteq\bigcup_{x=(ab)z}H_{(ab)z}=H_x$$

Thus

$$(GoH)oG\subseteq H.$$

Conversly, assume that: $GoHoG\subseteq H.$ Let $a,x,b\in G$, such that, $(ax)b=x$. Then

$$H_{(ax)b} \supseteq (GoHoG)_{(ax)b} = \bigcup_{(ax)b=pq} \left\{(G\circ H)_p \cap G_q\right\}$$
$$= \bigcup_{(ax)b=pq} \left\{ \bigcup_{p=yx} \left\{G_y \cap H_x\right\} \cap G_q \right\}$$
$$= \bigcup_{(ax)b=pq} \left\{ \bigcup_{p=yx} \left\{[0,1] \cap H_x\right\} \cap [0,1] \right\}$$
$$= \bigcup_{(ax)b=pq} \left\{ \bigcup_{p=yx} \left\{H_x\right\} \cap [0,1] \right\}$$
$$= \bigcup_{(ax)b=pq} \left\{ \bigcup_{p=yx} \left\{H_x\right\} \right\} = H_x.$$

Thus
$$\mathrm{H}_{(ax)b} \supseteq \mathrm{H}_x.$$

6. HESITANT FUZZY QUASI IDEALS IN AG-GROUPOIDS

Definition 3.20: A hesitant fuzzy set H of G is called a hesitant fuzzy quasi ideal of G if the following condition is valid:
$$(H\circ G)\cap(G\circ H)\subseteq H.$$

Theorem 3.21: Let $\phi \neq \mathrm{A} \subseteq G.$ Then A is a quasi-ideal of G if and only if the characteristic hesitant fuzzy set $\left[H_{\mathrm{A}}\right]$ is a hesitant fuzzy quasi-ideal of G .

Proof: Suppose that A is a quasi-ideal of G . Let $x\in G$, if $x\in \mathrm{A}.$ Then
$$([H_{\mathrm{A}}]\circ \mathrm{G})\cap(\mathrm{G}\circ[H_{\mathrm{A}}])_x \subseteq [0,1]= [H_{\mathrm{A}}]_x.$$
If $x \notin \mathrm{A},$ then
$$\left[H_{\mathrm{A}}\right]_x = \phi \subseteq ([H_{\mathrm{A}}]\circ \mathrm{G})\cap(\mathrm{G}\circ[H_{\mathrm{A}}])_x$$
$$\Rightarrow ([H_{\mathrm{A}}]\circ \mathrm{G})\cap(\mathrm{G}\circ[H_{\mathrm{A}}])_x \subseteq [H_{\mathrm{A}}]_x.$$
Conversly, assume that $\left[H_{\mathrm{A}}\right]$ is a quasi-ideal on G . Let x be an element of $(H\circ G)\cap(G\circ H)$. Then $ca = x = bd,\ \forall\ a,b\in G$ and $c,d\in \mathrm{A}.$ Then by defination we have
$$[H_{\mathrm{A}}]_x \supseteq (([H_{\mathrm{A}}]\circ \mathrm{G})\cap((\mathrm{G}\circ[H_{\mathrm{A}}]))_x$$
$$= ([H_{\mathrm{A}}]\circ \mathrm{G})_x \cap (\mathrm{G}\circ[H_{\mathrm{A}}])_x$$
$$= \left(\bigcup_{x=uv}\{[H_{\mathrm{A}}]_u \cap G_v\}\right)\cap\left(\bigcup_{x=uv}\{[H_{\mathrm{A}}]_u \cap G_v\}\right)$$
$$= \left(\bigcup_{x=uv}\{[H_{\mathrm{A}}]_u\}\right)\cap\left(\bigcup_{x=uv}\{G_v\}\right)=[0,1].$$
and so $x\in \mathrm{A}$. Thus $\mathrm{A}G\cap G\mathrm{A}\subseteq \mathrm{A}$ and hence A is a quasi-ideal of G .

REFERENCES

1. Akram, M., N. Yaqoob and M. Gulistan, Cubic KU-subalgebras, International Journal of Pure and Applied Mathematics, 89(5), 659-665, 2013.
2. Aslam, M., S. Abdullah, B. Davvaz and N. Yaqoob, Rough M-hypersystems and fuzzy M-hypersystems in Γ-semihypergroups, Neural Computing and Applications, 21(1), 281-287, 2012.
3. Faisal, N. Yaqoob and A. B. Saeid, Some results in bipolar-valued fuzzy ordered AG-groupoids, Discussiones Mathematicae - General Algebra and Applications, 32, 55-76, 2012.
4. Faisal, N. Yaqoob and K. Hila, On fuzzy (2,2)-regular ordered Γ-AG**-groupoids, UPB Scientific Bulletin, Series A: Applied Mathematics and Physics, 74(2), 87-104, 2012.
5. Faisal, N. Yaqoob and A. Ghareeb, Left regular AG-groupoids in terms of fuzzy interior ideals, Afrika Matematika, 24(4), 577-587, 2013.
6. Gulistan, M., N. Yaqoob and M. Shahzad, A note on Hv-LA-semigroups, UPB Scientific Bulletin, Series A: Applied Mathematics and Physics, 77(3), 93-106, 2015.
7. Gulistan, M., M. Shahzad and N. Yaqoob, On $(\in,\in\vee q_k)$ -fuzzy KU-ideals of KU-algebras, Acta Universitatis Apulensis, 39, 75-83, 2014.
8. Jun, Y. B. and S. Z. Song, Hesitant fuzzy set theory applied to filters in MTL- algebras, Honam Mathematical Journal, 36(4), 813-830, 2014.
9. M. A. Kazmi and M. Naseerudin, On LA-semigroup, Aligarh Bulletin of Mathematics, 8, 1-7, 1972.
10. Kazim, M. A. and M. Naseerudin, On almost semigroups, Aligarh Bulletin of Mathematics, 2, 1-7, 1972.
11. M. Khan, Structural properties of LA-semigroups, M.phil. Dissertation, Quaid-i-Azam University, Islamabad (2003).
12. Khan, M., V. Amjid, G. Zaman and N. Yaqoob, Characterizations of ordered Γ-Abel-Grassmann's groupoids, Discussiones Mathematicae - General Algebra and Applications, 34(1) (2014) 55-73.
13. Khan, M., T. Aziz and N. Yaqoob, Generalized interval-valued fuzzy right ideals of Abel Grassmann's groupoids, Analele Universităţii din Oradea - Fascicola Matematică, 21(1) (2014) 153-160.
14. Khan, M., Y. B. Jun, M. Gulistan and N. Yaqoob, The generalized version of Jun's cubic sets in semigroups, Journal of Intelligent and Fuzzy Systems, 28, 947-960, 2015.
15. Khan, M., Y. B. Jun and T. Mahmood, Generalized fuzzy interior ideals in Abel-Grassmann's groupoids, International Journal of Mathematics and Mathematical Sciences, Article ID 838392, 14 pages, 2010.
16. Khan, M., and M.N.A. Khan, On fuzzy Abel Grassmann's Groupoids, Advances in Fuzzy Mathematics, 5(3), 349-360, 2010.

17. Khan, M., K. P. Shum and M.F. Iqbal, Minimal ideals of Abel Grassmann's Groupoids, International Journal of Pure and Applied Mathematics, 83(1), 121-135, 2013.
18. Khan, M., F. Smarandache and S. Anis, Characterizations of left regular ordered Abel-Grassmann groupoids, International Journal of Algebra, 5(11), 109-110, 2015.
19. Mushtaq, Q., and M. Khan, Direct product of Abel Grassmann's Groupoids, Journal of Interdisciplinary Mathematics, 11, 461-467, 2008.
20. Mushtaq, Q., and M.S Kamran, On LA-semigroups with weak associative law, Science Khyber, 2(1)(1989)69-71.
21. Torra, V., Hesitant fuzzy sets, International Journal of Intelligent Systems, 25, 529-539, 2010.
22. Yaqoob, N., and M. Gulistan, Partially ordered left almost semihypergroups, Journal of the Egyptian Mathematical Society, 23, 231-235, 2015.
23. Yaqoob, N., M. Aslam, B. Davvaz and A. Ghareeb, Structures of bipolar fuzzy Γ-hyperideals in Γ-semihypergroups, Journal of Intelligent and Fuzzy Systems, 27(6), 3015-3032, 2014.
24. Yaqoob, N., and M. Aslam, Generalized rough approximations in Γ-semihypergroups, Journal of Intelligent and Fuzzy Systems, 27(5), 2445-2452., 2014.
25. Yaqoob, N., Interval-valued intuitionistic fuzzy ideals of regular LA-semigroups, Thai Journal of Mathematics, 11(3), 683-695, 2013.
26. Yaqoob, N., P. Corsini and F. Yousafzai, On intra-regular left almost semihypergroups with pure left identity, Journal of Mathematics, Article ID 510790, 10 pages, 2013.
27. Yaqoob, N., R. Chinram, A. Ghareeb and M. Aslam, Left almost semigroups characterized by their interval valued fuzzy ideals, Afrika Matematika, 24(2) 231-245, 2013.
28. Yaqoob, N., M. Akram and M. Aslam, Intuitionistic fuzzy soft groups induced by (t,s)-norm, Indian Journal of Science and Technology, 6(4), 4282-4289, 2013.
29. Yaqoob, N., and M. Aslam, Yao’s generalized rough approximations in left almost semigroups, Annals of Fuzzy Mathematics and Informatics, 3(1), 119-131, 2012.
30. Yousafzai, F., N. Yaqoob and A. Zeb, On generalized fuzzy ideals of ordered AG-groupoids, International Journal of Machine Learning and Cybernetics, in press, doi: 10.1007/s13042-014-0305-6.
31. Zadeh, L. A, Fuzzy sets, Information and Control, 8(3), 338-353, 1965.
32. Zenab, R., Some studies in fuzzy AG-groupoids, M.phil. Dissertation, Quaid-i-Azam University, Islamabad, 2009.