MODIFIED GOLBABAI AND JAVIDI'S METHOD (MGJM) FOR SOLVING NONLINEAR FUNCTIONS WITH CONVERGENCE OF ORDER SIX

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ABSTRACT: *In this paper, we describe the modified Golbabai and Javidi's method (MGJM) for solving nonlinear functions and analyzed. The modified Golbabai and Javidi's method (MGJM) has convergence of order 6 and efficiency index* $6^{1/3} \approx 1.8171$ *. The modified Golbabai and Javidi's method (MGJM) converges faster than Newton's method. Halley's method and Householder method (HHM). The comparison tables demonstrate the faster convergence of modified Golbabai and Javidi's method (MGJM).*
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INTRODUCTION

The problem, to recall, is solving equations in one variable. We are given a function f and would like to find atleast one solution of the equation $f(x)=0$. Note that, we do not put any restrictions on the function f; we need to be able to evaluate the function; otherwise, we cannot even check that a given $x=\xi$ is true, that is $f(r)=0$. In reality, the mere ability to be able to evaluate the function does not suffice. We need to assume some kind of "good behavior". The more we assume, the more potential we have, on the one hand, to develop fast iteration scheme for finding the root. At the same time, the more we assume, the fewer the functions are going to satisfy our assumptions! This is a fundamental paradigm in numerical analysis.

We know that one of the fundamental algorithm for solving nonlinear equations is so-called fixed point iteration method [1].

In the fixed-point iteration method for solving nonlinear equation $f(x)=0$, the equation is usually rewritten as

$$x=g(x), \qquad (1.1)$$

where

i) There exists $[a,b]$ such that $g(x)\epsilon[a,b]$ for all $x\epsilon\ [a,b]$,

ii) There exists (a,b) such that $|g'(x)|\le L<1$ for all $x\in(a,b)$.

Considering the following iteration scheme

$$x_{n+1}=g(x_n),\ n=0,1,2,\ldots \qquad (1.2)$$

and starting with a suitable initial approximation x_0, we build up a sequence of approximations, say $\{x_n\}$, for the solution of nonlinear equation, say ξ. The scheme will be converge to ε provided that

i) The initial approximation x_0 is chosen in the interval $[a,b]$,

ii) $|g'(x)|<1$ for all x ϵ [a,b],

iii) $a\le g(x)\le b$ for all $x\epsilon[a,b]$,

it is well known that the fixed point method has first order convergence. By using Taylor expansion, expanding $f(x)$ about the point x_k such that

$$f(x)= f(x_k)+(x-x_k)\ f'(x_k)+(x-x_k)^2 f''(x_k)+\ldots \qquad (*)$$

if $f'(x_k)\neq 0$, we can evaluate the above expression as follows:

$$f(x_k)+(x-x_k)\ f'(x_k)=0$$

if we choose x_{k+1} the root of equation, then we have

$$x_{k+1}= x_k-\frac{f(x_k)}{f'(x_k)} \qquad (1.3)$$

This is so-called the Newton's method (NM) [2,3,4] for root-finding of nonlinear functions, which converges quadratically.

From (*) one can evaluate

$$x_{k+1}=x_k-\frac{2f(x_k)f'(x_k)}{2f'^2(x_k)-f(x_k)f''(x_k)} \qquad (1.4)$$

This is so-called the Halley's method (HM) [5,6,7] for root-finding of nonlinear functions, which converges cubically.

The iterative methods with higher-order convergence are presented in some literature [12-18]. In [18], Householder gives an iterative method, called Householder's method (HHM) provisionally, which is expressed as

$$x_{k+1}=x_k-\frac{f(x_k)}{f'(x_k)}\left(1+\frac{f(x_k)f''(x_k)}{2f'^2(x_k)}\right) \qquad (1.5)$$

It is pointed out that HHM has cubic convergence. One can see easily that HHM requires the evaluation of first and second derivatives of the function $f(x)$.

Also Golbabai and Javidi give an iterative method, called Golbabai and Javidi's method (GJM) provisionally, which is expressed as

$$x_{k+1}=x_k-\frac{f(x_k)}{f'(x_k)}-\left(\frac{f(x_k)f''(x_k)}{2(f'^3(x_k)-f(x_k)f'(x_k)f''(x_k))}\right) \tag{1.6}$$

It is also pointed out that GJM has cubic convergence. One can see easily that GJM requires the evaluation of first and second derivatives of the function $f(x)$.

During the last century, the numerical techniques for solving nonlinear equations have been successfully applied (see, e.g. [2-18] and the references therein). T.J. McDouogall *et. Al.* [11] modified the Newton's method and their modified Newton's method have convergence of order(1+$\sqrt{2}$).

In this paper, we proposed a modified Golbabai and Javidi's method (MGHM) having convergence of order 6 extracted from Golbabai and Javidi's method [19] to solve nonlinear functions motivated by the technique of T.J. McDougall et al. [11]. The proposed modified Golbabai and Javidi's method (MGJM) applied to solve some problems in order to assess its validity and accuracy.

MAIN RESULTS

Let $f: X \to \mathbb{R}, X \subset \mathbb{R}$ is a scalar function then by using Taylor expansion, expanding $f(x)$ about the point x_k, we obtain the Golbabai and Javidi's method as follows:

$$x_{k+1}=x_k-\frac{f(x_k)}{f'(x_k)}-\left(\frac{f(x_k)f''(x_k)}{2(f'^3(x_k)-f(x_k)f'(x_k)f''(x_k))}\right).$$

By following the approach of T.J McDougall *et al.* [11], we develop modified Golbabai and Javidi's method (MGJM) as follows:

Initially, we choose two starting points x and x^*. Then, we set $x^*=x$, our modified Golbabai and Javidi's method (MGJM) that we examine herein is given

By

$$x_0^*=x_{0,}$$

$$x_1=x_0-\frac{f(x_0)}{f'(\frac{1}{2}[x_0+x_0^*])}-\left(\frac{f(x_0)f''(\frac{1}{2}[x_0+x_0^*])}{2(f'^3(\frac{1}{2}[x_0+x_0^*])-f(x_0)f'(\frac{1}{2}[x_0+x_0^*])f''(\frac{1}{2}[x_0+x_0^*]))}\right), \tag{2.1}$$

Which implies (for $k \geq 1$)

$$x_k^*=x_k-\frac{f(x_k)}{f'\left(\frac{1}{2}[x_{k-1}+x_{k-1}^*]\right)}-\left(\frac{f(x_k)f''\left(\frac{1}{2}[x_{k-1}+x_{k-1}^*]\right)}{2\left(f'^3\left(\frac{1}{2}[x_{k-1}+x_{k-1}^*]\right)-f(x_k)f'\left(\frac{1}{2}[x_{k-1}+x_{k-1}^*]\right)f''\left(\frac{1}{2}[x_{k-1}+x_{k-1}^*]\right)\right)}\right), \tag{2.2}$$

$$x_{k+1}=x_k-\frac{f(x_k)}{f'\left(\frac{1}{2}[x_k+x_k^*]\right)}-\left(\frac{f(x_k)f''\left(\frac{1}{2}[x_k+x_k^*]\right)}{2\left(f'^3\left(\frac{1}{2}[x_k+x_k^*]\right)-f(x_k)f'\left(\frac{1}{2}[x_k+x_k^*]\right)f''\left(\frac{1}{2}[x_k+x_k^*]\right)\right)}\right),$$

These are the main steps of our modified Golbabai and Javidi's method (MGJM).

The Value of x_2 is calculated from x_1 using $f(x_1)$ and the values of first and second derivatives of $f(x)$ evaluated at $\frac{1}{2}(x_1+x_1^*)$ (which is more appropriate value of the derivatives to use than the one at x_1), and this same value of derivatives is re-used in the next predictor step to obtain x_3^*. This re-use of the derivatives means that the evaluations of the starred values of x in (2.2) essentially come for free, which then enables the more appropriate value of the derivatives to be used in the corrector step (2.3).

Convergence Analysis

In this section we consider the convergence criteria of modified Golbabai and Javidi's method (MGJM).

Theorem 3.1. Let $f: X \subset R \to R$ for an open interval X and consider that the nonlinear equation $f(x)=0$ has a simple root $\alpha \epsilon X$, where $f(x)$ be sufficiently smooth in the neighborhood of α; then the convergence order of modified Golbabai and Javidi's method (MGJM) is at least six.

Proof: If α is the root and e_n be the error at nth iteration, than $e_n = x_n-\alpha$ and using Taylor series expansion, we have

$$f(x_n)= e_nf'(\alpha)+\frac{e_n^2}{2!}f^n(\alpha)+\frac{e_n^3}{3!}f^n(\alpha)+\frac{e_n^4}{4!}f^n(\alpha)\ldots\ldots.\frac{e_n^7}{7!}f^n(\alpha)+O(e_n^8), \tag{3.1}$$

$$f(x_n)=f'(\alpha)+[e_n+c_2e_n^2+c_3e_n^3+c_4e_n^4+\cdots+c_7e_n^7+O(e_n^8)], \tag{3.2}$$

Table 1. Comparison of efficiency of various methods

Method	Number of function or derivatives evaluations	Efficiency index
Newton, quadratic	2	$2^{\frac{1}{2}} \approx 1.4142$
Cubic Methods	3	$3^{\frac{1}{3}} \approx 1.4422$
Kuo's 5th order	4	$5^{\frac{1}{4}} \approx 1.4953$
Kou's 6th order	4	$6^{\frac{1}{4}} \approx 1.5651$
Secant	1	$0.5(1+\sqrt{5}) \approx 1.6180$
Jarratt's 4th order	3	$4^{\frac{1}{3}} \approx 1.5874$
MGJM	3	$6^{\frac{1}{3}} \approx 1.8171$

Table 2. Comparison of NM, HM, HHM and MGJM

Method	N	N_f	$f(x_{n+1})$	x_{n+1}
$f(x) = x^3 + 4x^2 - 25$				
$x_0 = 3.5$				
Newton's Method	7	14	$1.220281e-12$	
Halley's Method	4	12	$5.668049e-38$	2.035268481182001658568408945504
HHM	5	15	$1.741099e-86$	
Modified GJM	3	9	$2.545448e-46$	
$x_0 = 3$				
Newton's Method	6	12	$3.758013e-35$	
Halley's Method	4	12	$2.058131e-49$	2.035268481182001658568408945504
HHM	4	12	$3.063496e-38$	
Modified GJM	3	9	$3.768940e-48$	
$f(x) = x^2 - e^x - 3x + 2$				
$x_0 = 0.8$				
Newton's Method	5	10	$3.885907e-52$	
Halley's Method	4	12	$7.637654e-67$	0.257530285439860760455367304937
HHM	4	12	$3.712419e-68$	
Modified GJM	3	9	$2.082037e-42$	
$x_0 = -1$				
Newton's Method	6	12	$1.095467e-52$	
Halley's Method	4	12	$2.592457e-72$	0.257530285439860760455367304937
HHM	4	12	$2.862783e-56$	
Modified GJM	3	9	$7.353293e-39$	
$f(x) = x^2 + \sin\left(\frac{x}{5}\right) - \frac{1}{4}$				
$x_0 = 1$				
Newton's Method	7	14	$1.417760e-56$	
Halley's Method	4	12	$4.877772e-36$	0.409992017989137131621258376499
HHM	5	15	$8.105106e-85$	
Modified GJM	4	12	$5.510800e-76$	
$f(x) = xe^{x^2} - \sin(2x)^2 + 2\cos x + 5$				
$x_0 = -1.5$				
Newton's Method	7	14	$3.555074e-43$	
Halley's Method	4	12	$3.593061e-43$	-1.212245968111684115122854274201
HHM	5	15	$6.120645e-75$	
Modified GJM	3	9	$4.689731e-32$	

$$f'(x_n) = f'(\alpha) + [1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + \dots + 7c_7e_n^6 + O(e_n^7)] \quad (3.3)$$

$$f''(x_n) = f'(\alpha) + [1 + 6c_3e_n + 12c_4e_n^2 + \dots + O(e_n^8)] \quad (3.4)$$

Where $c_k = \frac{1}{k!}\frac{f^{(k)}(\alpha)}{f'(\alpha)}, k = 2,3,4 \dots . and\ e_n = x_n - \alpha$.

Now using (3.1-3.4) in (1.6), we get

$$e_{n-1} = -\frac{1}{72}\frac{(f'''(\alpha))^2 f''(\alpha)}{f''((\alpha))^3}e_n^6 + \dots .$$

Hence the convergence order of modified Golbabai and Javidi's method (MGJM) is at least six.

Comparison of efficiency index

Weerakoon and Fernando [12], Homeier [13] and Frontini, and Sormani [14] have presented numerical methods having cubic convergence. In each iteration of these numerical methods three evaluations are required of either the function or its derivative. The best way of comparing these numerical methods is to express the of convergence per function or derivative evaluation, the so-called "efficiency" of the numerical method. On this basis, the Newton's method has an efficiency of $2^{\frac{1}{2}} \approx 1.4142$, the cubic convergence methods have an efficiency of $3^{\frac{1}{3}} \approx 1.4422$.

Kuo [15] has developed several methods that each require two function evaluations and two derivative evaluations and these methods achieve an order of convergence of either five or six, so having efficiencies of $5^{\frac{1}{4}} \approx 1.4953$ and $6^{\frac{1}{4}} \approx 1.5651$ respectively. In these Kuo's methods the denominator is a linear combination of derivatives evaluated at different values of x, so that, when the starting value of x is not close to the root, this denominator may go to zero and the methods may not converge. Of the four 6^{th} order methods suggested in Kuo [15], if the ratio of function's derivatives at the two value of x differ by a factor of more than three, then the method gives an infinite change in x. That is, the derivatives at the predictor and corrector stages can both be the same sign, but if their magnitudes differ by more than a factor of three, the method does not converge.

Jarrat [16] developed a 4^{th} order method that requires only one function evaluation and two derivative evaluations, and similar 4^{th} order method have been described by Soleymani et al. [17]. Jarrat's method is similar to those of Kuo's methods in that if the ratio of derivatives at the predictor and corrector steps exceeds a factor of three, the method gives an infinite change in x. Jarratt's methods in similar to those of Kuo in that if the ratio of the derivatives at the predictor and corrector steps exceeds a factor of three, the method gives an infinite change in x. While the modified Golbabai and Javidi's method (MGJM) has an efficiency of $6^{\frac{1}{3}} \approx 1.8171$ larger than the efficiencies of all methods discussed above. The efficiencies of the methods we have discussed are summarized in Table 1 given below.

Applications

In this section we included some nonlinear functions to illustrate the efficiency of our developed modified Golbabai and Javidi's method (MGJM). We compare the MGJM with Newton's method, Halley's method and HHM as shown in Table 2 below.

Table 2 below shows the numerical comparisons of Newton's method, Halley's method, HHM and modified Golbabai and Javidi's method (MGJM). The columns represent the number of iterations N and the number of functions or derivatives evaluations N_f required to meet the stopping criteria, and the magnitude $|f(\mathrm{x})|$of $f(\mathrm{x})|$ at the final estimate x_n.

CONCLUSION

A modified Golbabai and Javidi's method (MGJM) for solving nonlinear functions has been established. We can concluded from tables (1,2) that

1. The modified Golbabai and Javidi's method (MGJM) has an efficiency of $6^{\frac{1}{3}} \approx 1.8171$ larger than the efficiencies of all methods discussed in Table 1.
2. The modified Golbabai and Javidi's method (MGJM) has convergence of order six.
3. By using some examples the performance of modified Golbabai and Javidi's method (MGJM) is also discussed. The modified Golbabai and Javidi's method (MGJM) is performing very well in comparison to HHM, Halley's method and Newton's method as discussed in Table 2..

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