FULLY DEVELOPED LIQUID LAYER FLOW OVER CONVEX CORNER INCLUDING SURFACE TENSION EFFECTS

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ABSTRACT: *In past the free surface flows gain much importance in the research area. There are many practical problems related to the free surface such as river flows, flow out of a teapot etc. The main concern of this paper is to study fully developed liquid layer flow including surface tension effects. In the direction of flow we encounter a convex corner. Assuming upstream flow to be fully developed we get exact solutions far upstream from half Posiuelle flow. Keeping α very small, the pressure term then obtained using the interaction law. The comparison can be made with the numerical results.*

Key words: Liquid Layer flow, Convex Corner, Pressure Distribution.

1. INTRODUCTION:

In this paper, our problem is to study the effects of surface tension on a liquid layer flow of a free surface confronting a convex corner. Free surface flows are very common to various practical problems for example; river flows, waterfalls and flow from a teapot etc. There arise many questions like what is the effect of boundary layer on the free surface? How does the hindrance effect the fluid flow? Also the scaling is carried out by different physical phenomena's. The basic aim is to visualize free surface flows with high Reynolds number considering the surface tension effects. Gajjar worked on the problem of liquid layer flow passing through a convex corner while neglecting the surface tension effects [1]. He dealt the problem in analytic as well as numeric way. In this particular scenario, surface tension effects are considered.

Merkin and Smith work on boundary layer flow near sharp edges [5,6]. Stewartson has also contributed to this work by studying the behavior of boundary layers on external and internal flows [2]. In 1904, Prandtl gave the concept of boundary layer theory which was a remarkable step in the history of fluid Mechanics [3]. This theory was valid for large Reynolds number. Prandtl divided the flow region into two parts, one is the inviscid flow region and the other is thin liquid layer near the boundary where the viscous forces are non-negligible.

Using double deck structure and asymptotic techniques leads us to the deduction of interaction law which shows the relation among pressure distribution and displacement of the boundary layer. i.e., $p = -sA - (1 - 1/C)A''$. Where "s" is the scaled measure of angle of and "C" is Capillary number.

2. MODELING OF THE PROBLEM:

We consider our flow to be two-dimensional, steady, laminar and fully developed. Taking coordinate system to be (x, y). Fluid is moving along the direction of x-axis with constant density ρ and viscosity μ under the effects of gravity along a wide flat plate. The plate is inclined at angle α to the horizontal. Also $y = h(x)$ is the unperturbed depth of liquid layer. The related velocities are (U_B, V_B) as depicted in figure 1. There exist surface tension effects between the fluid and surroundings denoted by σ as it was ignored by Gajjar[1].

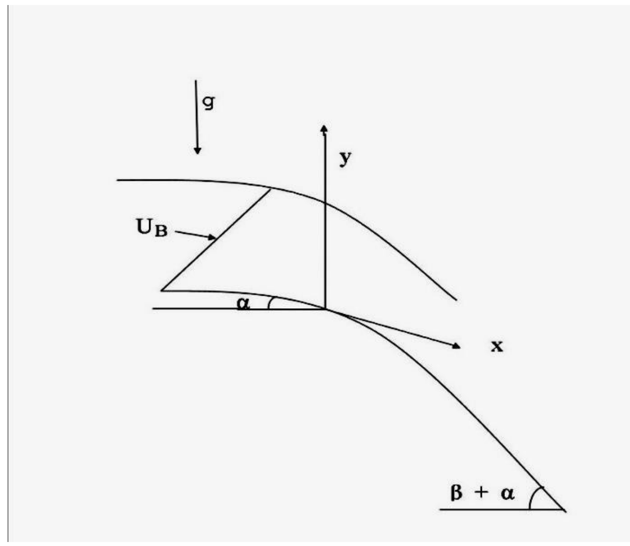

Figure 1: Basic geometry

GOVERNING EQUATIONS:

The basic flow far upstream is given as [4]

$$U_B = \frac{g \sin\beta}{2\nu}(2h_0 y - y^2), \quad V_B = 0 \ldots (3.1)$$

$$P_B = p_{atm} - \rho g \cos\beta\,(y - h_0) \ldots (3.2)$$

The related continuity equation and Navier Stokes equation for the flow problem is considered. To make dimensionless form, the distance is scaled by unperturbed length h_0. The governing equations are

$$u_x + v_y = 0 \ldots (3.3a)$$

$$uu_x + vu_y = -p_x + \frac{1}{R}(u_{xx} + u_{yy}) + \frac{2}{R} \ldots (3.3b)$$

$$uv_x + vv_y = -p_y + \frac{1}{R}(v_{xx} + v_{yy}) - \frac{2}{R}\cot\beta \ldots (3.3c)$$

Along with the boundary conditions of no slip, kinematic condition and the stress of normal and tangent balance that is

$$u = 0, \; v = 0 \text{ at } y = 0 \ldots (3.4a)$$

$$v = uh_x \ldots (3.4b)$$

$$(1 - h_x^2)(u_y + v_x) + 4h_x v_y = 0 \ldots (3.4c)$$

$$\frac{(1 + h_x^2)}{(1 - h_x^2)} v_y + \frac{1}{2}R(p'_{atm} - p) = \frac{h_{xx}}{2C(1 + h_x^2)^{3/2}} \ldots (3.4d)$$

There appear important dimensionless parameters. Here R is the Reynolds number which shows the ratio between inertial to viscous forces and C is the capillary number and is ratio between viscous to capillary forces. If U_C is the characteristic velocity and ν is the kinematic viscosity then both parameters are given by

$$R = \frac{U_C h_o}{\nu}, \qquad C = \frac{U_C \mu}{\sigma}.$$

3. EXPANSIONS:

Let $\bar{X}=\epsilon x$ be the scaled coordinate and $y = 1 + \epsilon^2 \eta + \cdots$ be the free surface. Where $y = 0(1)$

$$u \sim U_B + \epsilon^2 u_1 + o(\epsilon^4) \ldots (4.1a)$$

$$V \sim \epsilon^3 v_1 + o(\epsilon^5) \ldots (4.1b)$$

$$p \sim \epsilon^2 \bar{s}(1-y) + \epsilon^4 p_1 + o(\epsilon^6) \ldots (4.1c)$$

We need a viscous layer of thickness $O(\epsilon^2)$ and therefore $y = \epsilon^2 \bar{Y}$, $\bar{Y} \sim O(1)$

$$u \sim \epsilon^2 \bar{U}_1 + \cdots (4.2a)$$

$$v \sim \epsilon^5 \bar{V}_1 + \cdots (4.2b)$$

$$p \sim \epsilon^2 \bar{s} + \epsilon^4 (-s\bar{Y} + \bar{P}_1) + \cdots (4.2c)$$

After applying these expansions into (3.4a-3.4d) we have

$$\bar{U}_{1\bar{X}} + V_{1\bar{Y}} = 0 \ldots (4.3a)$$

$$\bar{U}_1 \bar{U}_{1\bar{X}} + \bar{V}_1 \bar{U}_{1\bar{Y}} = -P_{1\bar{X}} + \bar{U}_{1\bar{Y}\bar{Y}} \ldots (4.3b)$$

$$\bar{P}_{1\bar{Y}} = 0 \ldots (4.3c)$$

And the boundary conditions are

$$\bar{U}_1 = \bar{V}_1 = 0 \text{ at } \bar{Y} = -\alpha \bar{F}(\bar{X}) \ldots (4.4a)$$

$$\bar{U}_1 \to \bar{Y} + \bar{A}(\bar{X}) \quad \text{as} \quad \bar{Y} \to \infty \ldots (4.4b)$$

$$\bar{U}_1 \to \bar{Y} \quad \text{as} \quad \bar{X} \to -\infty \ldots (4.4c)$$

$$\bar{P}_1 = \bar{P}(\bar{X}) \ldots (4.4d)$$

To normalize we apply Prandtl transformation, so we get the following set of equations

$$U_X + V_Y = 0 \ldots (4.5a)$$

$$UU_X + VU_Y = -P_X + U_{YY}, \quad P = P(X) \ldots (4.5b)$$

$$U = V = 0 \quad \text{on} \quad Y = 0 \ldots (4.5c)$$

$$U \to Y + A - \alpha F(X) \text{ as } Y \to \infty \ldots (4.5d)$$

$$U \to Y \text{ as } X \to -\infty \ldots (4.5e)$$

$$P = -sA - \left(\frac{C-1}{C}\right) A'', \quad \eta = -A \ldots (4.5f)$$

4. LINEARIZED SOLUTIONS:

To obtain analytic results we choose α to be very small. Suppose

$$U = Y + \alpha \hat{U} + 0(\alpha^2)$$

$$V = \alpha \hat{V} + \cdots, \quad A = \alpha \hat{A} + \cdots, \quad P = \alpha \hat{P} + \cdots.$$

Substituting into the previous equations (4.5a-4.5f) and neglecting $O(\alpha^2)$ terms, we have set of linear equations for $\hat{U}, \hat{V}, \hat{P}$. The equations were then solved by using Fourier Transform which is defined by

$$p^*(w) = \int_{-\infty}^{\infty} p(X) e^{-iwx} dX,$$

We acquire following expressions

$$\hat{A}^*(w) = \frac{F^*(w)}{1 - \left(\frac{iw}{\theta}\right)^{\frac{1}{3}} \left(a - \left(1 - \frac{1}{C}\right)\left(\frac{w}{\theta}\right)^2\right)} \ldots (5.1a)$$

$$\hat{P}^*(w) = -\left(s - \left(1 - \frac{1}{C}\right) w^2\right) \hat{A}^* \ldots (5.1b)$$

$$\hat{\tau}_l^* = (\hat{U}_y^*)_{y=0} = \frac{Ai(0)}{Ai'(0)} (iw)^{2/3} \hat{P}^*(w) \ldots (5.1c)$$

Using inverse Fourier transformation in (5.1a-5.1c) and substituting $iw/\theta = z$, we find

$$\hat{A}(X) = \frac{1}{2\pi i \theta} \int_{-\infty}^{\infty} \frac{e^{\theta X z} \, dZ}{Z^2 \left[1 - Z^{1/3} \left\{a + \left(1 - \frac{1}{C}\right) Z^2\right\}\right]},$$

$$\hat{P}(X) = -s\tilde{A} - \left(1 - \frac{1}{C}\right)\left(\frac{\theta}{2\pi i}\right) \int_{-\infty}^{\infty} \frac{e^{\theta X Z} dZ}{\left[1 - Z^{1/3} \left\{a + \left(1 - \frac{1}{C}\right) Z^2\right\}\right]},$$

$$\hat{\tau}_l = \frac{3A_i(0)}{\theta^{7/3}} \left(\frac{-i}{2\pi \theta^{1/3}} \int_{-\infty}^{\infty} \frac{e^{\theta X z} \, dZ}{Z^{4/3} \left[1 - Z^{1/3} \left\{a + \left(1 - \frac{1}{C}\right) Z^2\right\}\right]} - \left(1 - \frac{1}{C}\right)\left(\frac{i\theta^{5/3}}{2\pi}\right) \int_{-\infty}^{\infty} \frac{Z^{2/3} e^{\theta X z} \, dZ}{\left[1 - Z^{1/3} \left\{a + \left(1 - \frac{1}{C}\right) Z^2\right\}\right]} \right).$$

There are exactly three roots, one is real and the other two are complex conjugates of each other. We can write as

$$q_1, q_{2,3} = re^{\pm i\phi}$$

So the path of integration is taken to be L which is at the right of imaginary axis. Also take a cut along negative real axis.

The pressure distribution for $\boldsymbol{X < 0}$,

When $\boldsymbol{C > 1}$

$$\tilde{P}(X) = \frac{-3\theta e^{\theta X q_1}}{{q_1}^{5/3} \left(a + 7\left(1 - \frac{1}{C}\right) q_1^2\right)},$$

When $\boldsymbol{C < 1}$

$$\tilde{P}(X) = -s\tilde{A} + \frac{6\theta \left(1 - \frac{1}{C}\right) r^{2/3} e^{\theta X r \cos\phi} \left\{a \cos\left(\frac{2\phi}{3} + \theta X r \sin\phi\right) + 7r^2 \left(1 - \frac{1}{C}\right) \cos\left(\theta X r \sin\phi - \frac{4\phi}{3}\right)\right\}}{a^2 + 14ar^2\left(1 - \frac{1}{C}\right)\cos\phi + 49\left(1 - \frac{1}{C}\right)^2 r^4},$$

When $\boldsymbol{C = 1}$

$$\tilde{P}(X) = \frac{-3\theta e^{\theta X q_1}}{a{q_1}^{5/3}}.$$

For $\boldsymbol{X > 0}$,

When $\boldsymbol{C > 1}$

$$\tilde{P}(X) = -s\tilde{A} + \left(1 - \frac{1}{C}\right)\left(\frac{\sqrt{3}\theta}{2\pi}\right) \int_0^\infty \frac{e^{-\theta X t} t^{1/3} \left(a + \left(1 - \frac{1}{C}\right) t^2\right) dt}{\left[1 - t^{1/3}\left(a + \left(1 - \frac{1}{C}\right) t^2\right) + t^{2/3}\left(a + \left(1 - \frac{1}{C}\right) t^2\right)^2\right]} + \frac{6\theta \left(1 - \frac{1}{C}\right) r^{2/3} e^{\theta X r \cos\phi} \left\{a \cos\left(\frac{2\phi}{3} + \theta X r \sin\phi\right) + 7r^2 \left(1 - \frac{1}{C}\right) \cos\left(\theta X r \sin\phi - \frac{4\phi}{3}\right)\right\}}{a^2 + 14ar^2\left(1 - \frac{1}{C}\right)\cos 2\phi + 49\left(1 - \frac{1}{C}\right)^2 r^4},$$

When $\boldsymbol{C < 1}$

$$\tilde{P}(X) = -s\tilde{A} + \left(1 - \frac{1}{C}\right)\left(\frac{\sqrt{3}\theta}{2\pi}\right) \int_0^\infty \frac{e^{-\theta X t} t^{1/3} \left(a + \left(1 - \frac{1}{C}\right) t^2\right) dt}{\left[1 - t^{1/3}\left(a + \left(1 - \frac{1}{C}\right) t^2\right) + t^{2/3}\left(a + \left(1 - \frac{1}{C}\right) t^2\right)^2\right]},$$

When $\boldsymbol{C = 1}$

$$\tilde{P}(X) = -s\tilde{A}.$$

RESULTS:

Fig 2 below, graph for linearized solution of pressure has been obtained using matlab, taking $C > 1$ and $s = 0.1$ we have

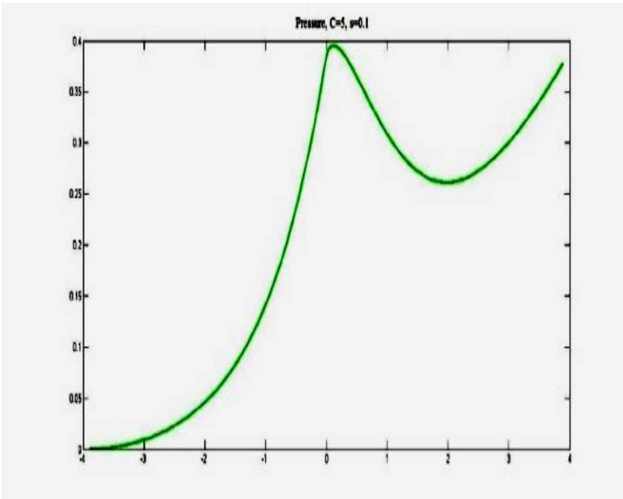

Fig. 2

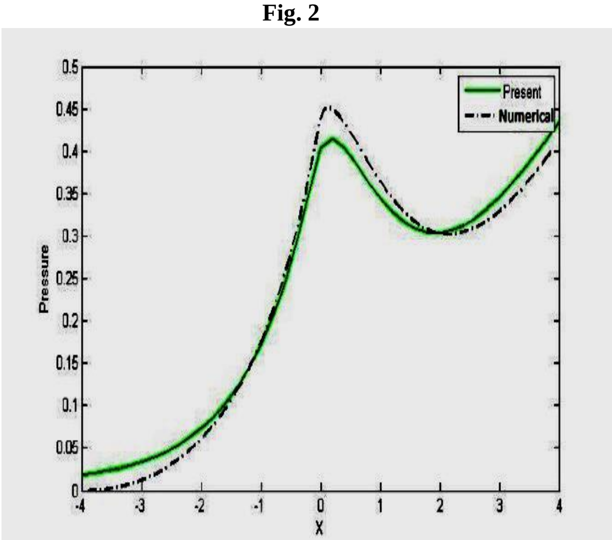

Fig. 3

COMPARISON:

For the accuracy in Fig 3, we compared our results with the numerical results. Assuming C=5 and s=0.1.

CONCLUSION:

The presence of surface tension effects in the flow field leads us to the interaction law

$$P = -sA - \left(1 - \frac{1}{C}\right)A''$$

The term C has much importance. For large capillary numbers the surface tension effects are negligible. At the same time it will be in good agreement with the results of Gajjar[1].

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