ANTIMAGIC TOTAL LABELING OF *M* ISOMORPHIC COPIES OF NON-ISOMORPHIC CYCLES

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ABSTRACT. *This paper deals with two types of graph labeling namely, super (a,d)-edge antimagic total labeling and super (a,d)-vertex antimagic total labeling. In this paper, we will prove that m isomorphic copies of non-isomorphic cycles are super (a,d)-edge antimagic total (EAT) and super (a,d)-vertex antimagic total (VAT) for $d = 1$.*

Keywords: super (a,d)-edge antimagic total labeling, super (a,d)-vertex antimagic total labeling, cycles.

1. INTRODUCTION

For undefined terms, see next section. In this section, we present the back ground and the motivation of the present study

1.1. History

A labeling of a graph is a mapping that takes the elements of the graph to the positive integers. If the domain is the vertex set or the edge set, the labeling is called vertex labeling or edge labeling respectively. Moreover, if the domain of the labeling is union of vertex set and edge set then it is called total labeling.

In 1960, on the motivation of the concept of magic square in number theory, Sedlacek [18,19] raised the problem to apply the magic ideas on graphs and introduced the notion of a magic labeling. Stewart [17] extended this study and proved that an n by n magic square in number theory corresponds to a super magic labeling of the complete bipartite graph $K_{n,n}$. In 1970's Kotzig and Rosa [9] defined the term of magic valuation for graphs. Later on, Ringel and Llado [16] introduced the same concept and called edge magic total labeling. Recently, Enomoto et al. [14], defined super edge magic total labeling. Later on, Simanjantuk et al. deffinition of (a; d)-edge antimagic total labeling [20,21]

Moreover, MacDougall et al. [22] defined the concept of vertex magic total labeling. As a natural extension of the vertex magic total labeling, Bača et al. in [1] defined the concept of (a,d)-vertex antimagic total labeling.

1.2. Motivation

Hartsfield and Ringel [8] proved that the path graphs P_n for $n \geq 3$, cycles, wheels and complete graphs K_n for $n \geq 3$ are antimagic. Further Bodendiek and Walther [4] proved that even cycles, paths of even order and stars all are not (a,d)-antimagic while c_{2k+1} is $(k + 2,1)$ antimagic.

Simanjuntak, Bertault and Miller [12] proved that C_n has $(2n+2,1)$ and $(3n + 2,1)$-EAT labeling, C_{2n} has $(4n + 2,2)$ and $(4n + 3,2)$-EAT labeling, C_{2n+1} has $(3n + 4,3)$ and $(3n + 5,3)$-EAT labeling.

Dafik, Miller, Ryan and Baca [5] investigated the Super edge antimagic labeling of isomorphic copies of cycles and path graphs. Some of their results are presented below:

- The graph mC_n has a super $(\frac{3mn+5}{2},2)$-EAT labeling if and only if m; n are odd and m; $n \geq 3$.
- The graph mC_n has a super $(2mn + 2,1)$-EAT labeling for every $m \geq 2$ and $n \geq 3$.

  Rahim, Ali and Javaid [10] investigated super (a,d)-EAT labeling and super (a,d)-VAT labeling of non-isomorphic copies of Harary graphs and cycles. Some of their results are presented as follows:
- For $m \geq 2$, $n_i \geq 3, i = 1,2,\dots,m$, $G \cong C_{n_1} \cup C_{n_2} \cup..\cup C_{n_m}$ admits super $(2\sum_{k=1}^{m} n_k + 2,1)$-EAT labeling.
- For $m \geq 2$, $n_i \geq 5, t_i \geq 2, i = 1,2,\dots,m$, $G \cong C_{t_1}^{n_1} \cup C_{t_2}^{n_2} \cup\cdot\cdot\cup C_{t_m}^{n_m}$ admits super $(2\sum_{k=1}^{m} n_k + 2,1)$-EAT labeling provided $n_i \neq 2t_i$.
- For $m \geq 2$, $n_i \geq 3, i = 1,2,\dots,m$, $G \cong C_{n_1} \cup C_{n_2} \cup..\cup C_{n_m}$ admits super $(3\sum_{k=1}^{m} n_k + 2,1)$-VAT labeling.
- For $m \geq 2$, $n_i \geq 5, t_i \geq 2, i = 1,2,\dots,m$, $G \cong C_{t_1}^{n_1} \cup C_{t_2}^{n_2} \cup\cdot\cdot\cup C_{t_m}^{n_m}$ admits super $(8\sum_{k=1}^{m} n_k + 3,1)$-VAT labeling provided $n_i \neq 2t_i$.

In this paper we prove that m isomorphic copies of non-isomorphic cycles are super $(a,1)$-EAT and super $(a,1)$-VAT.

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2. Preliminaries

In this section, we define basic definition and terminologies which are used frequently in the main results.

All the graphs in this paper are finite, simple and undirected. The graph G has the vertex set $V(G)$ and edge set $E(G)$. A (v,e)-graph G is a graph such that $|V(G)| = v$ and $|E(G)| = e$. For general graph theoretic notions see [6, 13].

2.1. Definition

A bijection $\lambda : V(G) \cup E(G) \rightarrow \{1,2,3,\dots,v + e\}$ is called (a,d)-EAT labeling of G if $\{\lambda(x) + \lambda(y) + \lambda(xy) : \in E(G)\}$ is equal to $\{a, a + d, a + 2d, \dots, a + (e-1)d\}$ for two integers $a > 0$ and $d \geq 0$. An (a,d)-EAT labeling is called super (a,d)-EAT labeling if $\lambda(E(G)) = \{v+1, v+2, v+3, \dots, v + e\}$. A graph that admits super (a,d)-EAT labeling is called super (a,d)-edge-antimagic graph.

2.2. Definition

A bijection $\lambda: V(G) \cup E(G) \rightarrow \{1, 2, 3 \ldots v + e\}$ is called (a, d)-VAT labeling of G if $\{\lambda(x) + \sum_{xy \in E(G)} \lambda(xy)\}$ is equal to $\{a, a + d, a + 2d, \ldots, a + (v-1)d\}$ for two integers a > 0 *and* d ≥ 0. An (a, d)-VAT labeling is called super (a, d)-VAT labeling if $\lambda(E(G)) = \{v+1, v+2, v+3, \ldots, ; v+e\}$. A graph that admits super (a, d)-VAT labeling is called super (a, d)-veretx-antimagic graph.

2.3. Definition

Let G be a regular graph. Let $\lambda: V(G) \cup E(G) \rightarrow \{1, 2, 3 \ldots v + e\}$ be an (a, d)-VAT labeling for G. Define a new labeling:

$\lambda': V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, v + e\}$

on G as follows:

$\lambda'(x) = v + e + 1 - \lambda(x)$; $x \in V(G)$

$\lambda'(xy) = v + e + 1 - \lambda(xy)$; $xy \in E(G)$

It is proved that λ' is also an (a', d)-VAT labeling for some a'. This new labeling λ' is called the dual of the labeling λ.

Remark: The same is true for (a, d)-EAT labeling [1].

3 MAIN RESULTS

This section covers the main results into two subsections which are given as follow:

3.1 Super (a, d)-EAT labeling of m isomorphic Copies of Non-Isomorphic Cycles

In this section we provide a labeling scheme for super $(a, 1)$-edge antimagic total labeling for disjoint union of m isomorphic copies of non- isomorphic cycles C_n.

Theorem 1. For every $m \geq 2$, $n_\beta \geq 3$, and $\beta = 1, 2, 3, \ldots, k$, where k is any positive integer, $G \cong mC_{n_1} \cup mC_{n_2} \cup, \ldots, \cup\, mC_{n_m}$ admits super $(2\, m \sum_{\beta=1}^{k} n_\beta + 2, 1)$-EAT labeling.

Proof. Let us denote the vertices and edges of G as follows:

$$V(G) = \{x_i^j \mid 1 \leq i \leq n_\beta;\ (\beta-1)\,m + 1 \leq j \leq \beta m\},$$

$$E(G) = \{x_i^j\, x_{i+1}^j \mid 1 \leq i \leq n_\beta - 1;\ (\beta-1)m + 1 \leq j \leq \beta m\} \cup \{x_i^j\, x_1^j \mid i = n_\beta;\ (\beta-1)m + 1 \leq j \leq \beta m\}.$$

Further, we have $v = m \sum_{\beta=1}^{k} n_\beta = e$.

$W = \{w(xy) : xy \in E(G)\} = \{a, a+1, a+2, \ldots, a + (m \sum_{\beta=1}^{k} n_\beta - 1)\}$ is the set of edge weights of G provided $a = 2\, m \sum_{\beta=1}^{k} n_\beta + 2$.

Now let α be a positive integer where $1 \leq \alpha \leq k$ such that when $\beta = k$ then $\alpha = k$.

Now, we define the labeling $\lambda: V(G) \cup E(G) \rightarrow \{1, 2, \ldots, 2m \sum_{\beta=1}^{k} n_\beta\}$ as follows:

for $1 \leq i \leq n_\beta$, and $((\beta - 1)m + 1) \leq j \leq \beta m$.

$$\lambda(x_i^j) = i + m \sum_{\beta=1}^{\alpha} n_{\beta-1} + (j - (\beta-1)m - 1)\, n_\beta$$

for $1 \leq i \leq (n_\beta - 1)$, and $((\beta - 1)m + 1) \leq j \leq \beta m$.

$$\lambda(x_i^j\, x_{i+1}^j) = 2v - i - m \sum_{\beta=1}^{\alpha} n_{\beta-1} + (1 + (\beta-1)m - j)\, n_\beta$$

for $i = n_\beta$ and $((\beta - 1)m + 1) \leq j \leq \beta m$.

$$\lambda(x_i^j\, x_1^j) = 2v - m \sum_{\beta=1}^{\alpha} n_{\beta-1} + (1 + (\beta-1)m - j)\, n_\beta$$

The edge weights of G constitute the sets:

for $1 \leq i \leq (n_\beta - 1)$, and $((\beta - 1)m + 1) \leq j \leq \beta m$.

$W_\lambda^1 = w(x_i^j\, x_{i+1}^j) = 2v + i + m \sum_{\beta=1}^{\alpha} n_{\beta-1} + (j - (\beta-1)m - 1)n_\beta + 1$

for $i = n_\beta$ and $((\beta - 1)m + 1) \leq j \leq \beta m$.

$W_\lambda^2 = w(x_i^j\, x_1^j) = 2v + m \sum_{\beta=1}^{\alpha} n_{\beta-1} + (j - (\beta - 1)m)n_\beta + 1$

Hence the sets $\cup_{t=1}^{2} W^t = \{2v + 2, 2v + 3, \cdots 3v + 1\}$ consist of consecutive integers. Thus, G is super $(2\, m \sum_{\beta=1}^{\alpha} n_\beta + 2, 1)$ EAT labeling. Note that the weight $2v + 2$ is attained by the edge $x_1^1\, x_2^1$.

Theorem 2. For every $m_\beta \geq 2$, $n_\beta \geq 3$, and $\beta = 1, 2, 3, \ldots, k$, where k is any positive integer, $G \cong m_1\, C_{n_1} \cup m_2\, C_{n_2} \cup, \ldots, \cup\, m_k\, C_{n_m}$ admits super $(2 \sum_{\beta=1}^{k} m_\beta\, n_\beta + 2, 1)$-EAT labeling.

Proof. Now let α be a positive integer where $1 \leq \alpha \leq k$ such that when $\beta = k$ then $\alpha = k$.

Let us denote the vertices and edges of G as follows:

$$V(G) = \{x_i^j \mid 1 \leq i \leq n_\beta;\ \sum_{\beta=1}^{\alpha} m_{\beta-1} + 1 \leq j \leq \sum_{\beta=1}^{\alpha} m_\beta\},$$

$$E(G) = \{x_i^j\, x_{i+1}^j \mid 1 \leq i \leq n_\beta - 1;\ \sum_{\beta=1}^{\alpha} m_{\beta-1} + 1 \leq j \leq \sum_{\beta=1}^{\alpha} m_\beta\} \cup \{x_i^j\, x_1^j \mid i = n_\beta;\ \sum_{\beta=1}^{\alpha} m_{\beta-1} + 1 \leq j \leq \sum_{\beta=1}^{\alpha} m_\beta\}.$$

Further, we have $v = \sum_{\beta=1}^{k} m_\beta\, n_\beta = e$.

$W = \{w(xy) : xy \in E(G)\} = \{a, a+1, a+2, \ldots, a + (\sum_{\beta=1}^{k} m_\beta\, n_\beta - 1)\}$ is the set of edge weights of G provided $a = 2 \sum_{\beta=1}^{k} m_\beta\, n_\beta + 2$.

Now, we define the labeling $\lambda: V(G) \cup E(G) \rightarrow \{1, 2, \ldots, 2 \sum_{\beta=1}^{k} m_\beta\, n_\beta\}$ as follows:

for $1 \leq i \leq n_\beta$, and $(\sum_{\beta=1}^{\alpha} m_{\beta-1} + 1) \leq j \leq \sum_{\beta=1}^{\alpha} m_\beta$.

$$\lambda(x_i^j) = i + \sum_{\beta=1}^{\alpha} m_{\beta-1}\, n_{\beta-1} + (j - \sum_{\beta=1}^{\alpha} m_{\beta-1} - 1)\, n_\beta$$

for $1 \leq i \leq (n_\beta - 1)$, and $(\sum_{\beta=1}^{\alpha} m_{\beta-1} + 1) \leq j \leq \sum_{\beta=1}^{\alpha} m_\beta$.

$$\lambda(x_i^j\, x_{i+1}^j) = 2v - i - \sum_{\beta=1}^{\alpha} m_{\beta-1}\, n_{\beta-1} + (1 + \sum_{\beta=1}^{\alpha} m_{\beta-1} - j)\, n_\beta$$

for $i = n_\beta$ and $(\sum_{\beta=1}^{\alpha} m_{\beta-1} + 1) \leq j \leq \sum_{\beta=1}^{\alpha} m_\beta$.

$$\lambda(x_i^j\, x_1^j) = 2v - \sum_{\beta=1}^{\alpha} m_{\beta-1}\, n_{\beta-1} + (1 + \sum_{\beta=1}^{\alpha} m_{\beta-1} - j)\, n_\beta$$

The edge weights of G constitute the sets:

for $1 \leq i \leq (n_\beta - 1)$, and $(\sum_{\beta=1}^{\alpha} m_{\beta-1} + 1) \leq j \leq$

$\sum_{\beta=1}^{\alpha} m_\beta$.

$W_\lambda^1 = w(x_i^j\, x_{i+1}^j) = 2v + i + \sum_{\beta=1}^{\alpha} m_{\beta-1}\, n_{\beta-1} + (j - \sum_{\beta=1}^{\alpha} m_{\beta-1} - 1)\, n_\beta + 1$

for $i = n_\beta$ and $(\sum_{\beta=1}^{\alpha} m_{\beta-1} + 1) \le j \le \sum_{\beta=1}^{\alpha} m_\beta$.

$W_\lambda^2 = w(x_i^j\, x_1^j) = 2v + \sum_{\beta=1}^{\alpha} m_{\beta-1}\, n_{\beta-1} + (j - \sum_{\beta=1}^{\alpha} m_{\beta-1})\, n_\beta + 1$

Hence the sets $\cup_{t=1}^{2} W^t = \{\, 2v + 2, 2v + 3, \cdots\ 3v + 1 \,\}$ consist of

consecutive integers. Thus, G is super $(2 \sum_{\beta=1}^{\alpha} m_\beta n_\beta + 2, 1)$ EAT

labeling. Note that the weight $2v + 2$ is attained by the edge $x_1^1\, x_2^1$.

3.2 Super (a,d)-VAT labeling of m isomorphic Copies of Non-Isomorphic Cycles

In this section we provide a labeling scheme for super $(a,1)$-vertex antimagic total labeling for disjoint union of m copies of non-isomorphic cycles C_n.

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Corollary 1. For every $m \ge 2$, $n_\beta \ge 3$, and $\beta = 1,2,3,\ldots,k$, where k is any positive integer, $G \cong mC_{n_1} \cup mC_{n_2} \cup,\ldots,\cup\ mC_{n_k}$ admits super $(3\, m \sum_{\beta=1}^{k} n_\beta + 2, 1)$-VAT labeling.

Proof. Using the dual of the labeling defined in theorem 1, the result follows.

Corollary 2. For every $m_\beta \ge 2$, $n_\beta \ge 3$, and $\beta = 1,2,3,\ldots,k$, where k is any positive integer, $G \cong m_1C_{n_1} \cup m_2C_{n_2} \cup,\ldots,\cup\ m_kC_{n_k}$ admits super $(3 \sum_{\beta=1}^{k} m_\beta\ n_\beta + 2, 1)$-VAT labeling.

Proof. Using the dual of the labeling defined in theorem 1, the result follows.

In reference to theorem 1 figure 1 given below shows a labeling scheme for $2C_3 \cup 2C_4 \cup 2C_5$ where $m = 2, n_1 = 3, n_2 = 4, n_3 = 5$ and $\beta = 1, 2, 3$

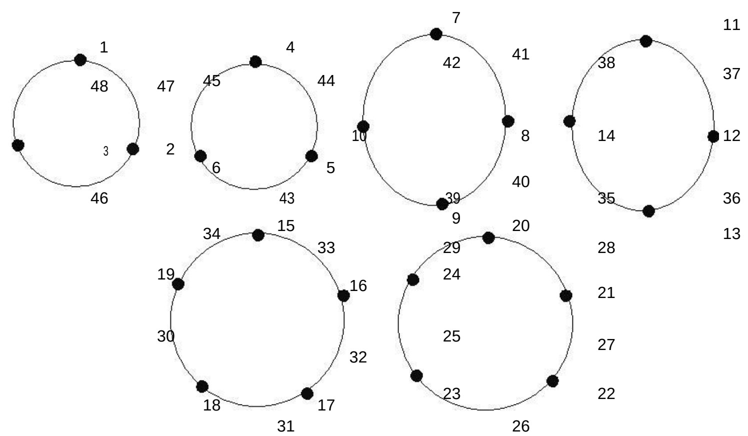

Figure 1. Super $(50,1)$-EAT Labeling of $2C_3 \cup 2C_4 \cup 2C_5$

4 CONCLUSION

In this paper, we have proved that for every $m \ge 2$, $n_\beta \ge 3$, and $\beta = 1,2,3,\ldots,k$, where k is any positive integer, $G \cong mC_{n_1} \cup mC_{n_2} \cup,\ldots,\cup\ mC_{n_k}$ and for $m_\beta \ge 2$, $n_\beta \ge 3$, and $\beta = 1,2,3,\ldots,k$, where k is any positive integer, $G \cong m_1C_{n_1} \cup m_2C_{n_2} \cup,\ldots,\cup\ m_kC_{n_k}$ admits super (a,d)-edge antimagic total labeling and super (a,d)-vertex antimagic total labeling for $d = 1$.

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