

# COMPUTATIONAL ANALYSIS OF TIME DEPENDENT HOLONOMIC CONSTRAINTS

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**ABSTRACT:** *The canonical method of time dependent holonomic constrained systems is discussed using the mathematical computational technique. The equations of motion are obtained as total differential equations in many variables, by applying the Euler-Lagrange equations subject to holonomic constraints; in this case the corresponding coordinates become arbitrary function of time. To explain the application of our formalism: the motion of a disk of mass m and radius R that is rolling down an inclined plane without slipping is discussed. The solution of the example is found in detail to be in exact agreement with the Euler-Lagrange equations when the time parameter ( $\mu$ ) is zero.*

**Keywords:** Constrained systems, Euler-Lagrange equations, Holonomic systems, Time dependent systems

## INTRODUCTION

The classical theory of mechanical systems with constraints was developed at the turn of the last century. In this theory the constraints are in the form

$F^k(q^i, \dot{q}^i, t) = 0$  which is dependent on the generalized coordinates  $q^i$ 's and the generalized velocities  $\dot{q}^i$ 's [1]. Constraints are restrictions that limit the motion of the particles of a system. The forces necessary to constrain the motion are said to be forces of constraint [2].

The constraints expressible as algebraic equations relating the coordinates of the particles and the time variable are called holonomic [3]. Holonomic constraints are relationships between the coordinates of the form [4, 5]:

$$f_k(x_A, t) = 0.$$

$$k = 1, 2, \dots, n$$

$$x = x_A(q_1, \dots, q_n)$$

In general the constraints can be time dependent and our notation above allow for this, holonomic constraints can be solved in terms of  $n$  generalized coordinates  $q_i$ , where  $i=1, 2, \dots, n$ .

We find the Euler-Lagrange equations for the systems in terms of the generalized coordinates. Because of constraints we need differential algebraic equations for modeling, their numerical analysis is available by a number of software packages. The equations of motion become standard ordinary differential equations, which could be integrated by any standard method. The simplest approach to solve a differential algebraic equation consists of integrating of equation [6]. This paper presents a computer-based method for formulation and efficient solution of the constrained differential equations of motion for mechanical systems, using a mathematica program which is a symbolic mathematical computation program, sometimes called a computer algebra program used in many scientific fields.

The holonomic constraint equations and differential equations of motion are written in terms of a generalized coordinates, to facilitate the general formulation of constraints and forcing functions. The main aim of this article is to show that various physically motivated formulations for the equations of motion of a time dependent holonomic constrained system may be understood as stabilization techniques.

This paper is organized as follows. Firstly, the Euler-Lagrange equations formulation of time dependent holonomic constrained systems is discussed. Then, an illustrative

example is discussed in detail. Finally, some concluding remarks are given.

## EULER-LAGRANGE EQUATION FORMULATION OF TIME DEPENDENT CONSTRAINED SYSTEMS

Many mechanical systems are subject to conservative forces which mean  $\vec{F} = -\nabla V$  that can be derived from a potential  $V(q)$  [7]. It is well known that the equations of motion of such systems take the form of natural Euler-Lagrange equations [8]:

$$\frac{d}{dt} \left( \frac{\partial L_o}{\partial \dot{q}_i} \right) - \left( \frac{\partial L_o}{\partial q_i} \right) = 0. \tag{1}$$

$i = 1, 2, \dots, n$  The traditional starting point for the modeling of a mechanical system is

Lagrangian function, not directly the equations of motion. Let  $q_i$  be the generalized coordinates in  $N$ - dimensional configuration space, we restrict our presentation to holonomic systems as explicit time dependence can always be treated by considering the time as additional coordinate in an extending configuration space. The Lagrangian is then a real- valued function  $L(q, \dot{q}, t)$  and the dynamic of a mechanical system described by it is given by the well known Euler-Lagrange equation. For natural mechanical system, the Lagrangian is of the form:

$L_o(q, \dot{q}) = T(\dot{q}) - V(q)$  where  $T$  stands for its kinetic energy and  $V$  stands for its potential energy [9]. The equation of motion for the harmonic oscillator for example is of this form [10]:

$$m\ddot{q} = -kq \text{ where}$$

$$V(q) = \frac{1}{2}kq^2$$

$$T(\dot{q}) = \frac{1}{2}m\dot{q}^2$$

But not all mechanical systems have an Euler-Lagrangian description, if the force  $\vec{F}$  is not a conservative force the equations of motion for holonomic constraints become of the general form [9]:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = \sum_k \lambda_k \frac{\partial f_k}{\partial q_i} \tag{2}$$

$$i=1, 2, \dots, n$$

$$k=1, 2, \dots, m$$

$\lambda_k$  are the Lagrange multipliers and they simply represent the forces of constraints. There is the same number of  $\lambda_k$  as the number of equations of constraints.

In equation (2) the LHS is the equation of motion for the unconstrained system and the RHS is the manifestation of the constraint forces in the system. This equation is more general than the natural Euler-Lagrange equation, because it still exhibits interesting property; one can conclude that the energy changes when the time passes. The systems which behave like this form are called time dependent systems and the Lagrangian of these systems is [11]:

$$L(q, \dot{q}, t) = L_o(q, \dot{q}, t)e^{\mu t} \tag{3}$$

where

$L_o$  : is called natural Lagrangian.  
 $\mu$ : is defined as time parameter,  $\mu > 0$ .

Taking the first partial derivative of the Lagrangian  $L(q, \dot{q}, t) = L_o(q, \dot{q}, t)e^{\mu t}$  with respect to the generalized coordinate  $\dot{q}$

$$\frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}} = \frac{\partial}{\partial \dot{q}} (L_o(q, \dot{q}, t)e^{\mu t}) = e^{\mu t} \frac{\partial L_o(q, \dot{q}, t)}{\partial \dot{q}} \tag{4}$$

The total time derivative of  $\partial L / \partial \dot{q}$  is:

$$\begin{aligned} \frac{d}{dt} \left( \frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}} \right) &= \frac{d}{dt} \left( \frac{\partial L_o(q, \dot{q}, t)e^{\mu t}}{\partial \dot{q}} \right) \\ &= e^{\mu t} \frac{d}{dt} \left( \frac{\partial L_o(q, \dot{q}, t)}{\partial \dot{q}} \right) \\ &\quad + \mu e^{\mu t} \frac{\partial L_o(q, \dot{q}, t)}{\partial \dot{q}} \end{aligned} \tag{5}$$

Also we differentiate the Lagrangian  $L(q, \dot{q}, t) = L_o(q, \dot{q}, t)e^{\mu t}$

with respect to the generalized coordinate  $q$

$$\frac{\partial L(q, \dot{q}, t)}{\partial q} = \frac{\partial}{\partial q} (L_o(q, \dot{q}, t)e^{\mu t}) = e^{\mu t} \frac{\partial L_o(q, \dot{q}, t)}{\partial q} \tag{6}$$

Finally, we differentiate our constraint function with respect to the generalized coordinate  $q$

$$\frac{\partial f}{\partial q} = \frac{\partial}{\partial q} (f(q, t)) \tag{7}$$

Now, we find the Euler-Lagrange equations to obtain the equations of motion for the system in terms of the coordinates  $q$  and  $\theta$ , by substituting equations (4-7) into equation (2).

The generalized momentum is [11]:

$$p_q = \frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}} = \frac{\partial}{\partial \dot{q}} (L_o(q, \dot{q}, t)e^{\mu t}) = (e^{\mu t}) \frac{\partial L_o(q, \dot{q}, t)}{\partial \dot{q}} \tag{8}$$

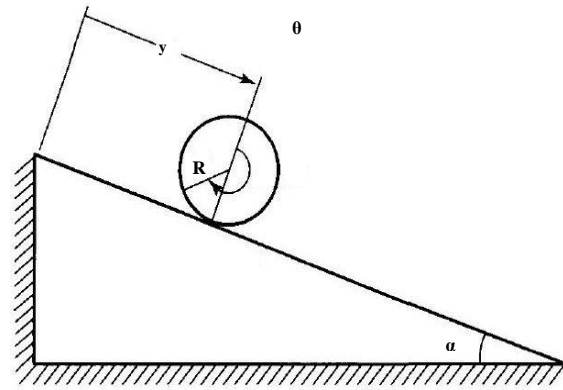
**ILLUSTRATIVE EXAMPLE**

**A disc Rolling Down an Inclined Plane**

To illustrate our work let us discuss the example that describes the motion of a disk of mass  $m$  and radius  $R$  that is rolling down an inclined plane without slipping, the Lagrangian is given by [12]:

$$L = \left( \frac{1}{2} m \dot{y}^2 + \frac{1}{4} m R^2 \dot{\theta}^2 + mgy \sin \alpha \right) e^{\mu t} \tag{9}$$

Where  $g$  is the acceleration due to the gravity, and  $\alpha$  is the angle of inclination. The equation of holonomic constraint that describes the relation between the coordinates ( $y$  and  $\theta$ ) is



$$f(y, \theta) = y - R\theta = 0 \tag{10}$$

The Euler-Lagrange equation for  $y$  coordinate is written as

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{y}} \right) - \left( \frac{\partial L}{\partial y} \right) = \lambda \left( \frac{\partial f}{\partial y} \right)$$

Then,

$$(m\mu\dot{y} + m\ddot{y} - mg \sin \alpha) e^{\mu t} = \lambda \tag{11}$$

Using the constraint

$$y = R\theta, \dot{y} = R\dot{\theta}, \ddot{y} = R\ddot{\theta} \tag{12}$$

Equation (11) can be written as

$$(\mu m R \dot{\theta} + m R \ddot{\theta} - m g \sin \alpha) e^{\mu t} = \lambda \tag{13}$$

multiplying equation (13) by  $R$ , we get:

$$(\mu m R^2 \dot{\theta} + m R^2 \ddot{\theta} - m R g \sin \alpha) e^{\mu t} = R \lambda \tag{14}$$

And the equation of motion of  $\theta$  coordinate is

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \left( \frac{\partial L}{\partial \theta} \right) = \lambda \left( \frac{\partial f}{\partial \theta} \right)$$

We find,

$$\left( \frac{1}{2} \mu m R^2 \dot{\theta} + \frac{1}{2} m R^2 \ddot{\theta} \right) e^{\mu t} = -R \lambda \tag{15}$$

Adding equation (14) to equation (15), we obtain:

$$\frac{3}{2} \mu m R^2 \dot{\theta} + \frac{3}{2} m R^2 \ddot{\theta} - m R g \sin \alpha = 0 \tag{16}$$

The computational solution of equation (16) takes the following form

$$DSolve[\frac{3}{2} m R^2 \mu \theta'[t] + \frac{3}{2} m R^2 \theta''[t] - m R g \sin \alpha == 0, \theta[t], t] \tag{17}$$

The output solution for this equation is

$$\{ \{ \theta[t] \rightarrow C[1] + t C[2] + \int_1^t \left( \frac{2 m R g \sin \alpha K[1]}{3 m R^2} - \mu \theta[K[1]] \right) dK[1] \} \} \tag{18}$$

The first derivative of  $\theta$  using equation (18)

is

$$D[C[1] + t C[2] + \int_1^t \left( \frac{2 m R g \sin \alpha K[1]}{3 m R^2} - \mu \theta[K[1]] \right) dK[1], t]$$

$$\theta'[t] = \frac{2 m R g \sin \alpha t}{3 m R^2} + C[2] - \mu \theta[t] \tag{19}$$

Taking the second derivative of  $\theta$  also

$$\begin{aligned} D \left[ \frac{2 m R g \sin \alpha t}{3 m R^2} + C[2] - \mu \theta[t], t \right] \\ \theta''[t] = \frac{2 m R g \sin \alpha}{3 m R^2} - \mu \theta'[t] \end{aligned} \tag{20}$$

Inserting the value of the first derivative and the second derivative of  $\theta$  into equation (13) the value of  $\lambda$  is

$$\lambda = \left( \begin{aligned} &\frac{2}{3} m g \mu \sin \alpha t + \mu m R C[2] - m R \mu^2 \theta[t] \\ &- \mu m R \theta'[t] - \frac{1}{3} m g \sin \alpha \end{aligned} \right) e^{\mu t} \tag{21}$$

And the accelerations

$$\ddot{\theta} = \frac{2}{3} \frac{g}{R} \sin \alpha - \mu \theta'[t] \tag{22}$$

$$\ddot{y} = \frac{2}{3} g \sin \alpha - R \mu \theta'[t] \tag{23}$$

In the limit  $\mu \rightarrow 0$ , we have

$$\lambda = \frac{-1}{3} mg \sin \alpha \tag{24}$$

And the disk accelerations are

$$\ddot{\theta} = \frac{2}{3} \frac{g}{R} \sin \alpha \tag{25}$$

$$\ddot{y} = \frac{2}{3} g \sin \alpha \tag{26}$$

These results are consistent with those results that we have obtained from Euler-Lagrange equations for natural Lagrangian.

Where the conjugate momenta are:

$$P_y = \frac{\partial L}{\partial \dot{y}} = m \dot{y} e^{\mu t} \tag{27}$$

$$P_\theta = \frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2} m R^2 \dot{\theta} e^{\mu t} \tag{28}$$

**CONCLUSION**

In this paper the non-natural Lagrangian with time dependent holonomic constraints is discussed. The starting point is the natural Euler-Lagrange equation

$$\frac{d}{dt} \left( \frac{\partial L_o}{\partial \dot{q}_i} \right) - \left( \frac{\partial L_o}{\partial q_i} \right) = 0.$$

We also saw that when some of the coordinates are dependent on each other and also time dependent which, appears using the parameter  $e^{\mu t}$ , the constraints can be included in the Euler-Lagrange equation in the following form,

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = \sum_k \lambda_k \frac{\partial f_k}{\partial q_i}$$

Euler-Lagrange equations formulation of time dependent holonomic constrained systems are investigated, the Lagrange multipliers and the solutions of the equations of motion are found for these systems using a mathematica program; we showed that using the example of a disk of mass  $m$  and radius  $R$  that is rolling down an inclined plane, the solution is found to be in exact agreement with Euler-Lagrange equation in the limit  $\mu \rightarrow 0$ .

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