AN ALGORITHM FOR CONSTRUCTING REDUCED ALTERNATING ACHIRAL KNOTS

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ABSTRACT: *Because of interesting and useful geometric as well as topological properties, alternating knots (links) were regarded to have an important role in knot theory and 3-manifold theory. Many knots with crossing number less than 10 are alternating. It was the properties of alternating knots that enable the earlier knot tabulators to construct tables with relatively few mistakes or omissions. Graphs of knots (links) have been repeatedly employed in knot theory. This article is devoted to establish relationship between knots and planar graphs. This relationship not only enables us to investigate the relationship among the regions and crossings of a reduced alternating achiral knot but also their relationship to the vertices, edges and faces of the corresponding planar graphs.. Consequently we have established an algorithm to construct reduced alternating achiral knots (links) through the planar graphs.*

Keywords*: Reidmeister Moves, LR-Graphs, Reduced Alternating Achiral knot, Planar Graphs, Equivalent* Knots
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INTRODUCTION

Knots have been introduced as early as is the story of Alexander and Gordian but never got scientific status till Nineteen Century. Once the Kelvin's Theory of Vortex Atoms was discarded, the knot theory merges as a part of Physics and then relegated to Mathematics. Mathematicians were perplexed at the seemingly unending number of ways a knot could be shaped and turned. Consequently, these give rise to the central problem of knot theory i.e., whether two knots (links) are equivalent or not especially whether a knot is equivalent to its mirror image or not. This was the motivation for much of the recent work in knot theory, which is devoted to search for invariants of knots. Reidemeister moves, tricoloring, knot polynomials(Alexander polynomial, Jones polynomial, Bracket polynomial, Homfly polynomial, Kauffman polynomial, etc.) are few examples. The study of invariants underwent in a kind of phase transition, which has linked knot theory to chemistry, molecular chemistry, mathematical physics, particles physics, polymer physics, statistical mechanics, fluid mechanics, kinematics, C^*-algebra, conformal field theory, crystallography, cryptography, graph theory, computer systems and networks, etc.In the recent past, biologists and chemists studying genetics discovered an exciting link of knot theory with DNA (genetic material of all cells, containing coded information about cellular molecules and processes) and synthetic chemistry [1,2]. DNA is just one application of knot theory, which presently is an area of intense mathematical activities worldwide.

Because of interesting and useful geometric as well as topological properties, alternating knots (links) were regarded to have an important role in knot theory and 3-manifold theory. Many knots with crossing number less than 10 are alternating. It were the properties of alternating knots that enable the earlier knot tabulators to construct tables with relatively few mistakes or omissions. It is conjectured that as the crossing number increases, the percentage of knots that are alternating goes to 0 exponentially quickly [3]. This article is devoted to establish relationship between knots and planar graphs. This relationship not only enables us to investigate the relationship among the regions and crossings of a reduced alternating achiral knot but also their relationship to the vertices, edges and faces of the corresponding planar graphs.. Consequently we have established an algorithm to construct reduced alternating achiral knots (links) through the planar graphs.

MATERIAL AND METHODS

Knots (links) will be confused with their projections. By the planar isotopy we mean the motion of the projection in the plane that preserves the graphical structure of the underlying universe. Two knots (links) in space can be deformed into each other (ambient isotopy) if and only if their projections can be transformed into one another by planar isotopy and the three Reidemeister moves. Two knots are equivalent (via Reidemeister moves) denoted by the symbol ~, if and only if (any of) their projections differ by a finite sequence of Reidemeister moves [4]. Graphs of knots (links) have been repeatedly employed in knot theory [6-8]. A knot (link) diagram can be considered as a planar graph with 4-valent vertices. We will call such a planar graph theuniverse of a knot (link). Kauffman [9] has established that "Universes of knots (links) are in one-to-one correspondence with planar graphs". Azram [10] has extended the same by establishing that the "Connected universes of knots (linked links) are in one-to-one correspondence with connected planar graphs".

For the construction of graph, shade (checker-board shading) the regions of knot (link) as black and white. Associate a pseudo graph to the knot(link) so that the vertices of the graph correspond to the black regions and the edges of the graph correspond to the crossings shared by the black regions. We will call this graph as the graph corresponding to black regions of the knot (link). See Figure 1.

Figure 1

The construction of the graph corresponding to white regions is exactly the same, all one needs to consider white regions instead of black regions. The LR-Graph is pseudo graph corresponding to the black (white) regions, where the edges are labeled as "L" or "R" depending on whether the upper string at the corresponding crossing falls on the left or on the right side when going from either black (white) region to the other adjacent black (white) region. If G be the graph corresponding to black (white) regions of a given knot (link) then by the "dual graph" of G, we mean the graph corresponding to the white (black) regions of the same knot (link). The unique choice of over/under structure for the crossings makes the knot as an alternating. LR-graph corresponding to a reduced alternating knot is always connected, planar, loop less, and bridgeless graph with all the labeling as "L" or "R". The result that "Connected universes of knots (linked links) are in one-to-one correspondence with connected planar graphs" can be generalized as "knots (linked links) are in one-to-one correspondence with connected planar LR-Graphs" [10]. This construction of LR-Graphs and vice versa does not required signed graphs and so is the orientation.

RESULTS AND DISCUSSION

Theorem 1.[11] The total number of regions and crossings in a reduced alternating achiral knot is always even.

Proof.Let K be a reduced alternating achiral knot with K^* as its mirror image. Let G_B and G_W be graphs corresponding to black and white regions of K respectively. Let G^*_B be the graph corresponding to the black regions of K^* . K being achiral implies G_B , G_W and G^*_B are equivalent via Reidemeister moves to each other's.

Without loss of generality, assume all the edges of G_B are labeled as "R". G_B and G_W are dual and K^* is mirror image of K, i.e., only the over-crossing are being changed to under-crossings and vice versa. Consequently, G^*_B and G_B are same except the labeling of the edges which implies that G_W can also be constructed from G^*_B in the same fashion as from G_B with the exception of labeling of edges. Hence, G^*_B and G_W are dual via construction but with the same labeling. Consequently G_B , G_W and G^*_B are dual to each otherexcept the labeling of edges. Therefore, G^*_B can be constructed by

(i) changing the labeling of the edges of G_B

(ii) By construction.

(i) Implies that G_B and G^*_B have same number of edges and vertices respectively.

(ii) Implies that number of vertices of G^*_B = number of regions of G_B and numbers of regions of G^*_B = number of vertices of G_B.

Hence number of black regions of K = number of vertices of G_B = number of regions of G^*_B = number of regions of G_B = number of vertices of G_W = number of white regions. Consequently, total number of regions of K is even.

Hence, total number of regions and crossings in a reduced alternating achiral knot is even. This completes the proof.

The regions and crossings correspond to the vertices and edges of the corresponding planar graph respectively. Important thing to note is that the numbers of edges in the planar graph that correspond to a reduced alternating achiral knot (link) are positive even ♦.

Theorem 2.[10] A reduced alternating achiral knot has the number of its black regions equal to the number of its white regions (B = W).

Proof. Let K be a reduced alternating achiral knot with K^* as its mirror image. K being achiral implies;

(1) $K \sim K^*$, $W(K) = W(K^*) = 0$,Also

(2) $f_{K^*}(A) = f_K(A^{-1})$, K being reduced alternating implies

(3) maxdeg$\langle K \rangle = V + 2(W-1)$, And

(4) mindeg$\langle K \rangle = -V - 2(B-1)$

Since $f_K = \alpha^{-W(K)}\langle K \rangle = -A^{-3W(K)}\langle K \rangle$. Therefore, using (1), (3) and (4) we get

(5) maxdeg$f_K = -3\,W(K) + V + 2(W-1) = V + 2(W-1)$

And

(6) mindeg$f_K = -3\,W(K) + V - 2(B-1) = -V - 2(B-1)$

Considering (2) and the fact that $K \sim K^*$ we get; $f_K(A) = f_K(A^{-1})$ which implies

(7) maxdegf_K= -min degf_K

Finally, using (5), (6) and (7) we get; V + 2 (B-1) = V + 2 (W-1) that is B = W. Hence, total number of regions and crossings in a reduced alternating achiral knot is even. This completes the proof ♦.

This means that the vertices of the planar graph that corresponds to a reduced alternating achiral knot (link) may be even or odd.

Theorem3.[11] Number of crossings in a reduced alternating achiral knot is two less that its number of regions.

Proof. Let K has 2p regions. Let G_B and G_W be the LR-Graphs corresponding to black and white regions of K respectively. Let

p =number of black regions of K= number of vertices of $G_B = V(G_B)$

q = number of edges of $G_B = E(G_B)$

r = number of regions (faces) of $G_B = r(G_B)$

By the Euler formula we have $p - q + r = 2$

Regarding the construction;

$V(G_W) = r(G_B)$ and $r(G_W) = V(G_B)$

K being reduced alternating achiral implies B = W. Therefore, $p = V(G_B) = B = W = V(G_W) = r(G_B) = r$. Consequently by Euler formula q = 2 (p-1).

Since the edges of G_B or G_W correspond to the crossings of a reduced alternating achiral K, therefore, the number of crossings of K is topological as well as ambient isotopy invariant. Hence, K has 2p-2 number of crossings. This completes the proof ♦.

This is a vital result that gives numerical relationshop among the vertices and edges of the planar graph corresponding to a reduced alternating achiral knot.

Theorem 4.[10] Let K be a reduced alternating knot such that; i. B = W ii.Two of its black regions (among 2n regions) share n-1 crossings. Then K is achiral.

Proof. Let K be a reduced alternating knot satisfying (i) and (ii). Let $G_B(G_W)$ be LR-Graph corresponding to black (white) regions of K. K being reduced implies $G_B(G_W)$ is loopless, bridgeless planar connected graph. K being alternating implies $G_B(G_W)$ have all the labeling as L or R. K has 2n regions and B = W implies K has n black regions, which imply that; $n = V(G_B) = V(G_W) = r(G_B)$ (construction).

Now, if $q = E(G_B)$ then using Euler formula, we have $q = n + r - 2$. In G_B, $n = r$, therefore, $q = 2n-2$. Hence, G_B has n vertices with 2n-2 edges. Consequences of condition (ii), two vertices of G_B share n-1 edges so the remaining n-1 edges are only sufficient to join each consecutive pair of vertices by one and only one edge, that is, G_B is n-gon with n-1 edges in common with a consecutive pair of vertices. It may be noted that any other choice will destroy the properties of G_B that G_B is loopless, bridgeless planar connected LR-Graph with 2n-2 edges. Hence, G_B is a special graph always isomorphic to its companion. In fact G_B is same as G_W if one ignores the planar isotopy and the labeling then the knot K that correspond to G_B is equivalent to its mirror image that correspond to G_W, that is, K is achiral. This completes the proof ♦.

The important observation in this theorem is the condition (ii) and conclusion.

Now observe the graphs in Figure 2 for n = 2, 3, and 4, which correspond to reduced (because there is no loop or bridge), alternating (because the labeling is either all R or L) achiral (because the dual graph is isomorphic and so is equivalent).

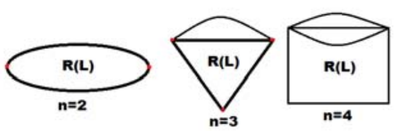

Figure 2

By a careful study, one can observe the unique construction of the graphs in Figure 2 which definitely correspond to a reduced alternating achiral knot (link). Consequently; we can write a short algorithm which enable us to construct graphs with n greater than or equal to 2 and subsequently reduced alternating knot or link.

ALGORITHM

- n = number of vertices where n is greater than or equal to 2 but finite
- Number of edges = 2n – 2.
- Construct n-gone which will occupy n edges out of 2n - 2.
- Put the remaining n-2 edges between any two adjacent vertices.
- Label each edge as L or R
- Construct the corresponding universe and then the knot or link which will be reduced alternating achiral because of the argument stated for graphs in Figure 2.

DILLEMA: By looking the graph resulting by the construction of the above algorithm, I suspect it is not possible to tell whether the result will be a knot or link and in case of a link then how many components.

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